EQUIDISTRIBUTION AND COUNTING FOR ORBITS OF GEOMETRICALLY FINITE HYPERBOLIC GROUPS

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Abstract. Let Q be a quadratic form of signature $(n,1)$, Γ a non-elementary discrete subgroup of $G = \mathrm{SO}_Q(\mathbb{R})$ and $w_0 \in \mathbb{R}^{n+1}$ a non-zero vector with the orbit $w_0\Gamma$ discrete. We introduce the notion of the Γ-skinning size $\mathrm{sk}_\Gamma(w_0)$ of w_0, and we compute an asymptotic formula (as $T \to \infty$) for the number of points in $w_0\Gamma$ of norm at most T, provided both the Bowen-Margulis-Sullivan measure m_Γ^{BMS} and the Γ-skinning size $\mathrm{sk}_\Gamma(w_0)$ are finite. For Γ geometrically finite, it is known that $|m_\Gamma^{\mathrm{BMS}}| < \infty$ by Sullivan and we give a criterion on the finiteness of $\mathrm{sk}_\Gamma(w_0)$: $\mathrm{sk}_\Gamma(w_0) < \infty$ if and only if either $\delta_\Gamma > 1$ or w_0 is not externally Γ-parabolic. We also prove new weighted equidistribution theorem for normal geodesic evolution of codimension one totally geodesic immersions. However the applicability of the basic result is not limited to geometrically finite groups

We use these results to count circles in the circle packing $\mathcal{P}$ of an ideal triangle of the hyperbolic plane $\mathbb{H}^2$ made by repeatedly inscribing the largest circles into the triangular interstices. We obtain that the number of circles in $\mathcal{P}$ of hyperbolic curvature at most T is asymptotic to $c \cdot T^{1.30568(8)}$ where $c = c(\mathbb{H}^2) > 0$ is an absolute constant.

Contents

1. Introduction

1.1. Let $Q(x_1, \cdots, x_{n+1})$ be a real quadratic form of signature $(n,1)$ for $n \geq 2$. For $m \in \mathbb{R}$, consider the level set V:

$$V := \{X \in \mathbb{R}^{n+1} : Q(X) = m\}.$$

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The variety V is a cone ($m = 0$) a one-sheeted hyperboloid ($m > 0$) or a two sheeted hyperboloid ($m < 0$) depending on the signature of m. Denote by G the identity component of the special orthogonal group

$$\mathrm{SO}_Q(\mathbb{R}) = \{g \in \mathrm{SL}_{n+1}(\mathbb{R}) : Q(g(X)) = Q(X)\}.$$

A discrete subgroup of a locally compact group with finite co-volume is called a lattice. For $v \in \mathbb{R}^{n+1}$ and a subgroup H of G, we denote by H_v the stabilizer of v in H.

Theorem 1.1 (Duke-Rudnick-Sarnak [9])**.** *Let Γ be a lattice in G and $w_0 \in \mathbb{R}^{n+1}$ a non-zero vector such that $w_0\Gamma$ is discrete. Suppose that Γ_{w_0} is a lattice in G_{w_0}. Then for any norm $\|\cdot\|$ on $\mathbb{R}^{n+1}$,*

$$\#\{w \in w_0\Gamma : \|w\| < T\} \sim \frac{\mathrm{vol}(\Gamma_{w_0}\backslash G_{w_0})}{\mathrm{vol}(\Gamma\backslash G)} \mathrm{vol}(B_T) \quad \text{as } T \to \infty$$

where $B_T := \{w \in w_0G : \|w\| < T\}$ and the volumes on G_{w_0}, G and $w_0G \simeq G_{w_0}\backslash G$ are computed with respect to invariant measures chosen compatibly.

Eskin and McMullen [11] gave a simpler proof of [9], based on the mixing property of the geodesic flow of a hyperbolic manifold with finite volume. This approach for counting via mixing was first used by Margulis's 1970 thesis [24].

The main goal of this paper lies in extending Theorem 1.1 to discrete subgroups Γ of infinite covolume in G. Let $\Gamma < G$ be a torsion-free discrete subgroup which is non-elementary, that is, Γ has no abelian subgroup of finite index. As G is isomorphic to the group of orientation preserving isometries of the hyperbolic space $\mathbb{H}^n$, Γ acts on $\mathbb{H}^n$ properly discontinuously. We denote by $0 \le \delta_\Gamma \le n-1$ the critical exponent of Γ and by $\{\nu_x : x \in \mathbb{H}^n\}$ a Γ-invariant conformal density of dimension δ_Γ on the geometric boundary $\partial(\mathbb{H}^n)$, which exists by the work of Patterson and Sullivan ([31], [41]). We denote by m_Γ^{BMS} the Bowen-Margulis-Sullivan measure on the unit tangent bundle $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ associated to $\{\nu_x\}$ (see Def. 3.3).

We set $V := w_0G$ and identify $\mathbb{H}^n$ and $\partial(\mathbb{H}^n)$ with a connected component of $\{Q(v) = -1\}$ and the set of lines $V_\infty := \{[v] = \mathbb{R}v : Q(v) = 0\}$ respectively. For $u \in \mathrm{T}^1(\mathbb{H}^n)$, we denote by $u^+ \in V_\infty$ the forward endpoint of the geodesic determined by u and by $\pi(u) \in \mathbb{H}^n$ the basepoint of u. For $x_1, x_2 \in \mathbb{H}^n$ and $\xi \in V_\infty$, let $\beta_\xi(x_1, x_2)$ denote the value of the Busemann function, i.e., the signed distance between horospheres based at ξ and passing through x_1 and x_2.

We denote by $Q(\cdot,\cdot)$ the bilinear-form associated to Q, and by $\mathbf{p}$ the canonical projection map $\mathrm{T}^1(\mathbb{H}^n) \to \mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$.

Definition 1.2. Define $\tilde{E}_{w_0} \subset \mathrm{T}^1(\mathbb{H}^n)$ as follows:

(1) For $Q(w_0) > 0$, let $\tilde{E}_{w_0}$ be the set of all unit normal vectors of the codimension one hyperbolic subspace $\{w \in \mathbb{H}^n : Q(w_0, w) = 0\}$.
(2) For $Q(w_0) = 0$, let $\tilde{E}_{w_0}$ be the set of all unit normal vectors of the horosphere $\{w \in \mathbb{H}^n : Q(w_0, w) = -1\}$ based at $[w_0] \in V_\infty$.

(3) For $Q(w_0) < 0$, let $\tilde{E}_{w_0}$ be the set of all unit vectors based at $\frac{w_0}{\sqrt{|Q(w_0)|}} \in \mathbb{H}^n$.

Define the following Borel measure on $\tilde{E}_{w_0}$:

$$d\mu^{\mathrm{PS}}_{\tilde{E}_{w_0}}(v) := e^{\delta_\Gamma \beta_{v^+}(o,\pi(v))} d\nu_o(v^+)$$

where $o \in \mathbb{H}^n$. This definition is independent of the choice of $o \in \mathbb{H}^n$ and by the Γ-invariance property of $\{\nu_x\}$, it induces a measure on $E_{w_0} = \mathbf{p}(\tilde{E}_{w_0})$, which we denote by $\mu^{\mathrm{PS}}_{E_{w_0}}$.

Definition 1.3 (The Γ-skinning size of w_0)**.** We define $0 \le \mathrm{sk}_\Gamma(w_0) \le \infty$ as follows:

$$\mathrm{sk}_\Gamma(w_0) = |\mu^{\mathrm{PS}}_{E_{w_0}}|.$$

Note that as Γ is non-elementary, $\delta_\Gamma > 0$ and $|m^{\mathrm{BMS}}_\Gamma| > 0$.

Theorem 1.4. *Let $\Gamma < G$ be a discrete subgroup with $|m^{\mathrm{BMS}}_\Gamma| < \infty$. Let $w_0 \in \mathbb{R}^{n+1}$ be a non-zero vector such that $w_0\Gamma$ is discrete and $\mathrm{sk}_\Gamma(w_0) < \infty$. Let $o \in \pi(\tilde{E}_{w_0})$ and G_o denote its stabilizer in G. Let $\|\cdot\|$ be a G_o-invariant norm on $\mathbb{R}^{n+1}$. Then*

$$\#\{w \in w_0\Gamma : \|w\| < T\} \sim \frac{|\nu_o| \cdot \mathrm{sk}_\Gamma(w_0)}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}_\Gamma| \cdot \|\check{w}_0\|^{\delta_\Gamma}} \cdot T^{\delta_\Gamma}, \tag{1.1}$$

where $\check{w}_0 \in \{Q = 0\}$ is such that $\mathbb{R}$-$\mathrm{span}\{o, \check{w}_0\} \ni w_0$ and $\check{w}_0$ is the Q-orthogonal projection of w_0 on $\mathbb{R}\check{w}_0$. Moreover, $\mathrm{sk}_\Gamma(w_0) > 0$ when $w_0\Gamma$ is infinite.

The description of the constant term changes if we do not put any restriction on the norm $\|\cdot\|$ (see Theorem 6.7). Stronger versions of Theorem 1.4 on the asymptotic number of points in $w_0\Gamma$ within a given sector (or cone) in V are obtained in Theorems 6.6 and 6.14.

Sullivan [41] showed that $|m^{\mathrm{BMS}}_\Gamma| < \infty$ when Γ is *geometrically finite*, i.e., when the unit neighborhood of its convex core[1] has finite volume. For instance, any discrete group admitting a finite sided polyhedron as a fundamental domain in $\mathbb{H}^n$ is geometrically finite.

We give a criterion on the finiteness of $\mathrm{sk}_\Gamma(w_0)$ for Γ geometrically finite. When $Q(w_0) > 0$, G_{w_0} is isomorphic to $\mathrm{SO}(n-1,1)$ and is the isometry group of the codimension one totally geodesic subspace $\tilde{S}_{w_0} = \{w \in \mathbb{H}^n : Q(w_0, w) = 0\}$ of $\mathbb{H}^n$. For $\xi \in \partial(\mathbb{H}^n)$, we denote by Γ_ξ the stabilizer of ξ in Γ and call ξ a *parabolic fixed point* of Γ if Γ_ξ is a parabolic subgroup (cf. Def. 3.1).

Definition 1.5. We say that $w_0 \in \mathbb{R}^{n+1}$ with $Q(w_0) > 0$ is *externally Γ-parabolic* if there exists a parabolic fixed point of Γ in the boundary of $\tilde{S}_{w_0}$ which is not fixed by any non-trivial element of Γ_{w_0}.

[1]The *convex core* C_Γ of Γ is defined to be the minimal convex set in $\Gamma\backslash\mathbb{H}^n$ which contains all geodesics connecting any two points in $\Lambda(\Gamma)$.

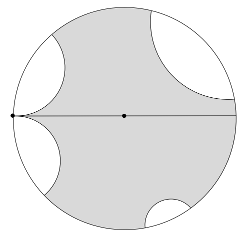

FIGURE 1. An externally Γ-parabolic vector

For $n = 2$, the externally Γ-parabolicity condition is equivalent to the geometric condition that at least one end of the geodesic $\tilde{S}_{w_0}$ goes into a cusp of a fundamental domain of Γ in $\mathbb{H}^2$ (see Fig. 1).

Theorem 1.6 (Criterion on the finiteness of $\mathrm{sk}_\Gamma(w_0)$)**.** *Let* Γ *be geometrically finite and* $w_0\Gamma$ *be discrete.*

(1) *If* $\delta_\Gamma > 1$, *then* $\mathrm{sk}_\Gamma(w_0) < \infty$.
(2) *If* $\delta_\Gamma \leq 1$, *then* $\mathrm{sk}_\Gamma(w_0) = \infty$ *if and only if* w_0 *is externally* Γ-*parabolic.*

Corollary 1.7. *Let* Γ *be geometrically finite and* $w_0\Gamma$ *discrete. If either* $\delta_\Gamma > 1$ *or* w_0 *is not externally* Γ*-parabolic, then* (1.1) *holds.*

Remark 1.8. (1) For geometrically finite Γ, if the Lebesgue volume of E_{w_0} is finite then $\mathrm{sk}_\Gamma(w_0)$ is finite (Corollary 1.13).

(2) If $\delta_\Gamma \leq 1$ and w_0 is externally Γ-parabolic, our preliminary study indicates that the asymptotic count should be of the order $T \log T$ if $\delta_\Gamma = 1$ and of the order T if $\delta_\Gamma < 1$, instead of T^{δ_Γ}.

(3) Theorem 1.4 is not limited to geometrically finite groups as Peigné [32] constructed a large class of geometrically infinite groups admitting a finite Bowen-Margulis-Sullivan measure. Note that $\mathrm{sk}_\Gamma(w_0)$ is finite whenever E_{w_0} is compact.

(4) When V is a two sheeted hyperboloid (i.e., $Q(w_0) < 0$), the orbital counting with respect to the hyperbolic metric balls was obtained by Lax and Phillips [23] for Γ geometrically finite with $\delta_\Gamma > (n-1)/2$, by Lalley [21] for convex cocompact subgroups and by Roblin [36] for all groups with finite Bowen-Margulis-Sullivan measure.

(5) When V is a light cone (i.e., $Q(w_0) = 0$) and Γ is geometrically finite with $\delta_\Gamma > (n-1)/2$, a version of Theorem 1.4 was obtained in [19].

1.2. **Weighted Equidistribution of expanding submanifolds.** We explain the main ergodic ingredients of a proof of Theorem 1.4. Let $\tilde{E} \subset \mathrm{T}^1(\mathbb{H}^n)$ be one of the following:

(1) an unstable horosphere
(2) the unit normal bundle of a codimension one totally geodesic subspace of $\mathbb{H}^n$

(3) the set of all outward normal vectors to a fixed hyperbolic sphere in $\mathbb{H}^n$.

Let Γ be a non-elementary discrete subgroup of the group $G = \mathrm{Isom}^+(\mathbb{H}^n)$ of orientation preserving isometries and $E := \mathbf{p}(\tilde{E})$ the image of $\tilde{E}$ under the projection $\mathbf{p} : \mathrm{T}^1(\mathbb{H}^n) \to \mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$. Choose a G-invariant conformal density $\{m_x : x \in \mathbb{H}^n\}$ on $\partial(\mathbb{H}^n)$ of dimension $(n-1)$.

We consider the following measures on E (cf. Def. 3.5):

$$d\mu_E^{\mathrm{Leb}}(v) = e^{(n-1)\beta_{v^+}(o,\pi(v))} dm_o(v^+), \quad d\mu_E^{\mathrm{PS}}(v) = e^{\delta_\Gamma \beta_{v^+}(o,\pi(v))} d\nu_o(v^+),$$

where $o \in \mathbb{H}^n$. Let m_Γ^{BR} denote the Burger-Roblin measure ([6], [36]) on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ associated to $\{\nu_x\}$ and $\{m_x\}$ (see Def. 3.3), and $\{g^t\}$ denote the geodesic flow on $\mathrm{T}^1(\mathbb{H}^n)$.

Theorem 1.9. *Suppose that $|m_\Gamma^{\mathrm{BMS}}| < \infty$. Let $E_0 \subset E$ be a Borel subset with $\mu_E^{\mathrm{PS}}(E_0) < \infty$ and $\mu_E^{\mathrm{PS}}(\partial(E_0)) = 0$. For $\psi \in C_c(\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n))$,*

$$(1.2)\quad e^{(n-1-\delta_\Gamma)t} \cdot \int_{E_0} \psi(g^t(v))\, d\mu_E^{\mathrm{Leb}}(v) \sim \frac{\mu_E^{\mathrm{PS}}(E_0)}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot m_\Gamma^{\mathrm{BR}}(\psi) \quad \text{as } t \to \infty.$$

In particular, when $|\mu_E^{\mathrm{PS}}| < \infty$, the result holds for $E_0 = E$.

When E is a horosphere and E_0 is bounded, Theorem 1.9 was obtained earlier by Roblin [36, P.52]. We were motivated to formulate and prove the result from an independent view point; our attention is especially on the case of $\pi(E)$ being totally geodesic immersion. This case involves many new features, observations, and applications. Note that when E_0 is unbounded, even when $\mu_E^{\mathrm{PS}}(E_0) < \infty$, one may still have $\mu_E^{\mathrm{Leb}}(E_0) = \infty$, and hence the proof involves greater care. The main key to our proof is the the transversality theorem 3.9, which was influenced by the work of Schapira [40]. The transversality theorem provides a precise relation between the transversal intersections of geodesic evolution of E_0 with a given piece of a weak stable leaf and the conditional of the m_Γ^{BMS} measure on that weak unstable piece.

In the case when both $\Gamma\backslash\mathbb{H}^n$ and E are of finite Riemannian volume, the measures m_Γ^{BMS} and m_Γ^{BR} are the projections of G-invariant measures to $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ and the measure μ_E^{PS} is the projection of $G_{\tilde{E}}$-invariant measure of $\tilde{E}$. In this case, Theorem 1.9 is due to Sarnak [39] for horocycles in $\mathbb{H}^2$ and Duke-Rudnick-Sarnak [9] and Eskin-McMullen [11] in general (see also the appendix of [17]).

In deducing Theorem 1.4 from Theorem 1.9, the standard techniques of orbital counting via equidistribution results require significant modifications due to the fact m_Γ^{BR} is not G-invariant; and this is achieved in section 6.

1.3. **On closedness of E and finiteness of μ_E^{PS}.** For a geometrically finite Γ, we have that m_Γ^{BMS} is finite. Next important condition for the application of Theorem 1.8 is to determine when μ_E^{PS} is finite. We denote by $G_{\tilde{E}}$ and $\Gamma_{\tilde{E}}$ the set-wise stabilizers of $\tilde{E}$ in G and in Γ, respectively.

Theorem 1.10. *Suppose that Γ is geometrically finite and that E is a closed subset of* $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$.

(1) *If $\pi(\tilde{E})$ is horospherical or a hyperbolic sphere, then* $\operatorname{supp}(\mu_E^{\mathrm{PS}})$ *is compact, and hence* $|\mu_E^{\mathrm{PS}}| < \infty$.
(2) *If $\pi(\tilde{E})$ is totally geodesic and $\delta_\Gamma > 1$ then $|\mu_E^{\mathrm{PS}}| < \infty$.*
(3) *If $\pi(\tilde{E})$ is totally geodesic and $\delta_\Gamma \le 1$, then for any parabolic fixed point $\xi \in \Lambda(\Gamma) \cap \partial(\pi(\tilde{E}))$, Γ_ξ is virtually cyclic, and it is either virtually contained in $\Gamma_{\tilde{E}}$ or virtually disjoint from $\Gamma_{\tilde{E}}$. If the former case occurs for every such ξ, then* $\operatorname{supp}(\mu_E^{\mathrm{PS}})$ *is compact and $|\mu_E^{\mathrm{PS}}| < \infty$, and if the latter case occurs for one such ξ, then $|\mu_E^{\mathrm{PS}}| = \infty$.*

The Theorem 1.6 is indeed deduced from its above geometric formulation.

The proof of Theorem 1.10 provides the following new result.

Theorem 1.11. *Let Γ be geometrically finite and $\tilde{S} \subset \mathbb{H}^n$ be a complete totally geodesic subspace. If $\Gamma\tilde{S}$ is closed in $\mathbb{H}^n$, then $\Gamma_{\tilde{S}}$ is a geometrically finite subgroup of* $\operatorname{Isom}(\tilde{S})$, *where $\Gamma_{\tilde{S}} = \{\gamma \in \Gamma : \gamma\tilde{S} = \tilde{S}\}$.*

As a byproduct of our study of transversal measures, we obtain the following interesting application:

Theorem 1.12. *Let Γ be a Zariski dense discrete subgroup of G. Assume that the Patterson-Sullivan density $\{\nu_x\}$ is atom-free. If $|\mu_E^{\mathrm{Leb}}| < \infty$ or $|\mu_E^{\mathrm{PS}}| < \infty$, then E is a closed subset of* $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$.

We remark that if $|m_\Gamma^{\mathrm{BMS}}| < \infty$, then the Patterson-Sullivan density is atom-free [36].

Another consequence of Theorems 1.10 and 1.12 is the following:

Corollary 1.13. *If Γ is geometrically finite and $|\mu_E^{\mathrm{Leb}}| < \infty$, then $|\mu_E^{\mathrm{PS}}| < \infty$.*

1.4. **The integrability of ϕ_0 and a characterization of a lattice.** Define $\phi_0 \in C(\Gamma\backslash\mathbb{H}^n)$ by

$$\phi_0(x) := |\nu_x| \quad \text{for } x \in \Gamma\backslash\mathbb{H}^n.$$

The function ϕ_0 is an eigenfunction of the hyperbolic Laplace operator with eigenvalue $-\delta_\Gamma(n-1-\delta_\Gamma)$ [41]. Sullivan [42] showed that if $\delta_\Gamma > (n-1)/2$, then $\phi_0 \in L^2(\Gamma\backslash\mathbb{H}^n)$ if and only if $|m_\Gamma^{\mathrm{BMS}}| < \infty$. The following theorem, which is a new application of Ratner's theorem [35], relates the integrability of ϕ_0 with the finiteness of $\operatorname{Vol}(\Gamma\backslash\mathbb{H}^n)$:

Theorem 1.14. *Let Γ be a non-elementary discrete subgroup of G. The following statements are equivalent:*

(1) $\phi_0 \in L^1(\Gamma\backslash\mathbb{H}^n)$;
(2) Γ *is a lattice in* G;
(3) $|m_\Gamma^{\mathrm{BR}}| < \infty$.

We remark that Γ being a lattice implies that $\delta_\Gamma = n-1$ and hence ϕ_0 is a constant function by the uniqueness of the harmonic function [44].

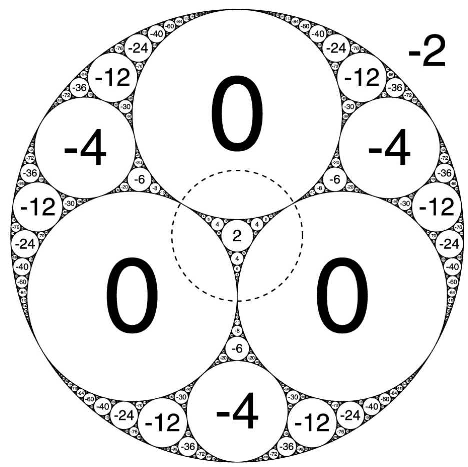

FIGURE 2. A hyperbolic Apollonian circle packing, reproduced from [11].

1.5. **Application to integral Apollonian circle packings.** We discuss an application of Theorem 1.4 to a circle packing problem. Given a set of four mutually tangent circles in the plane, we continue to repeatedly fill the interstices between mutually tangent circles with further tangent circles and obtain an infinite circle packing, called an *Apollonian circle packing*, say, $\mathcal{P}$ (cf. [16], [15], [38], [19]).

A natural question is to compute the asymptotic number of circles in $\mathcal{P}$ of radius at least $1/T$, as $T \to \infty$. There are three different notions of the *radius* for a given circle in the plane, in three different geometries: the Euclidean, the hyperbolic, and the spherical geometry. We find it more convenient to label the circles using the corresponding curvatures, rather than the radii.

By labeling each circle C in $\mathcal{P}$ by its *Euclidean* curvature $\mathrm{Curv}_E(C)$, that is, the reciprocal of its Euclidean radius, it is shown in [19] that for $\mathcal{P}$ bounded,

$$\#\{C \in \mathcal{P} : \mathrm{Curv}_E(C) < T\} \sim c \cdot T^{\alpha} \tag{1.3}$$

where α is the residual dimension of $\mathcal{P}$, that is, the Hausdorff dimension of the closure of $\mathcal{P}$, viewed as a union of countably many circles. The number α is independent of a packing $\mathcal{P}$ and is approximately 1.30568(8) according to McMullen [26]. The same type of asymptotic as (1.3) holds for any $\mathcal{P}$ with a bounded (Euclidean) period if we count circles only in a fixed bounded period of $\mathcal{P}$. In particular, any integral Apollonian packing, i.e., all curvatures being integral, has a bounded period.

We extend this result to spherical and hyperbolic Apollonian packings studied in [20] and [10]. Our approach in this paper also presents an alternative proof of (1.3).

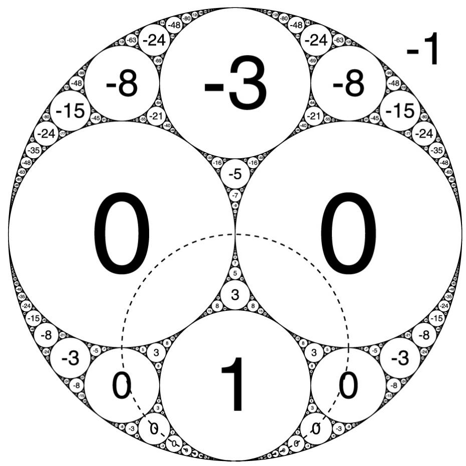

FIGURE 3. A hyperbolic Apollonian packing, reproduced from [11].

A hyperbolic Apollonian packing is an Apollonian circle packing $\mathcal{P}$ consisting of *hyperbolic circles*, that is, the image of hyperbolic circles of the two sheeted hyperboloids $\mathbb{H}^2_{\pm} = \{x_1^2 + x_2^2 - x_3^2 = -1\}$ via the stereographic projections from the both poles $(0, 0, \pm 1)$. We include degenerate hyperbolic circles, consisting of horocycles, the ideal boundary as well as the hyperbolic geodesics which intersect the ideal boundary at right angles.

The hyperbolic curvature of C of $\mathcal{P}$ is given by

$$\mathrm{Curv}_H(C) = \pm \coth r(C)$$

where $r(C)$ is the hyperbolic radius of C and the signature of the curvature is given positive if C is the projection of the upper sheet and negative otherwise. The hyperbolic Soddy-Gosset theorem (see [20, Thm. 6.1]) provides infinitely many integral hyperbolic circle packings. Denote by C_0 the ideal boundary and H the subgroup of Mobius transformations which preserve C_0. Denote by $\Gamma_{\mathcal{P}}$ the subgroup of Mobius transformations which preserve $\mathcal{P}$. $\Gamma_{\mathcal{P}}$ is known as the Apollonian group associated to $\mathcal{P}$. The works of [20] and [10] imply that for $\mathcal{P}$ integral, a hyperbolic period $\mathcal{P}_0$ exists, i.e., $\mathcal{P} = \cup_{\gamma \in \Gamma_{\mathcal{P}} \cap H} \gamma(\mathcal{P}_0)$, $(\Gamma_{\mathcal{P}} - H)\mathcal{P}_0 \cap \mathcal{P}_0$ is a union of finitely many circles and $\#\{C \in \mathcal{P}_0 : |\mathrm{Curv}_H(C)| < T\} < \infty$ for any $T > 1$. Since the absolute value of the curvature of a circle in $\mathcal{P}$ is preserved by the action of $\Gamma_{\mathcal{P}} \cap H$, we only count the circles in $\mathcal{P}_0$.

In Fig. 2, the circles inside the triangle formed by the circles of hyperbolic curvatures 2, 0, and 0 form a period $\mathcal{P}_0$ of $\mathcal{P}$, where $\Gamma_{\mathcal{P}} \cap H$ is generated by the inversion about the ideal circle C_0 (=the dotted circle) and a rotation of order 3 about the center of C_0. In Fig. 3, the circles inside a triangle formed by the circles of hyperbolic curvatures 3, 0, and 0, form a period of $\mathcal{P}$, where $\Gamma_{\mathcal{P}} \cap H$ is generated by the inversion in C_0 and a cyclic parabolic group fixing a point on C_0.

Theorem 1.15. *For any integral hyperbolic Apollonian packing $\mathcal{P}$ with a period $\mathcal{P}_0$ as above,*

$$\#\{C \in \mathcal{P}_0 : |\mathrm{Curv}_H(C)| < T\} \sim c_1 \cdot T^\alpha \quad \text{for some } c_1 > 0.$$

This theorem in particular applies to the circle packing of the ideal triangle of the Poincare disc, as illustrated by Fig. 2.

Theorem 1.16. *Let $\mathcal{P}$ be the circle packing of any ideal triangle of the hyperbolic plane $\mathbb{H}^2$ made by repeatedly inscribing the largest circles into the triangular interstices.*

Then the number of circles in $\mathcal{P}$ of (hyperbolic) curvature at most T is asymptotic to $c \cdot T^\alpha$ where $0 < c < \infty$ is an absolute constant independent of $\mathcal{T}$ and $\alpha = 1.30568(8)$ is the residual dimension of $\mathcal{P}$.

Since any two ideal triangles are isometric to each other, the constant c above does not depend on the choice of $\mathcal{T}$ and hence is a geometric invariant of $\mathbb{H}^2$.

A spherical Apollonian packing is defined similarly using the stereographic projection of the upper part of the unit sphere $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$ with the pole $(0, 0, -1)$. The spherical metric on $\mathbb{S}^2$ between two points on it is simply the angle between the rays connecting these points to the origin and the spherical curvature of a circle C of $\mathcal{P}$ is given by

$$\mathrm{Curv}_S(C) = \cot r(C)$$

where $r(C)$ is the spherical radius of C.

Theorem 1.17. *For any integral spherical Apollonian packing $\mathcal{P}$,*

$$\#\{C \in \mathcal{P} : \mathrm{Curv}_H(C) < T\} \sim c_2 \cdot T^\alpha \quad \text{for some } c_2 > 0.$$

In [29] and [30], we present applications of Theorem 1.10 to the question of counting circles in a given circle packing of the Euclidean plane, or of the unit sphere, invariant under geometrically finite Kleinian groups. This includes counting circles in various regions in Apollonian circle packings, Sierpinski curves, Schottky dances etc. Some of the main results in this paper are announced in [28].

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2. Transverse measures

2.1. Let $(\mathbb{H}^n, d)$ denote the hyperbolic n-space and $\partial(\mathbb{H}^n)$ its geometric boundary. Let G denote the identity component of the isometry group of $\mathbb{H}^n$. We denote by $\mathrm{T}^1(\mathbb{H}^n)$ the unit tangent bundle of $\mathbb{H}^n$ and by π the natural projection from $\mathrm{T}^1(\mathbb{H}^n) \to \mathbb{H}^n$. By abuse of notation, we use d to denote a left G-invariant metric on $\mathrm{T}^1(\mathbb{H}^n)$ such that $d(\pi(u), \pi(v)) = \min\{d(u_1, v_1) : \pi(u_1) = \pi(u), \pi(v_1) = \pi(v)\}$. For a subset A of $\mathrm{T}^1(\mathbb{H}^n) \cup \mathbb{H}^n \cup \partial(\mathbb{H}^n)$ and a

subgroup H of G, we denote by H_A the stabilizer subgroup $\{g \in H : g(A) = A\}$ of A in H.

Denote by $\{g^r : r \in \mathbb{R}\}$ the geodesic flow. For $u \in \mathrm{T}^1(\mathbb{H}^n)$, we set

$$u^+ := \lim_{r\to\infty} g^r(u) \quad \text{and} \quad u^- := \lim_{r\to-\infty} g^r(u)$$

which are the endpoints in $\partial(\mathbb{H}^n)$ of the geodesic defined by u. Note that $(g(u))^\pm = g(u^\pm)$ for $g \in G$. The map $\mathrm{Viz} : \mathrm{T}^1(\mathbb{H}^n) \to \partial(\mathbb{H}^n)$ given by $\mathrm{Viz}(u) = u^+$ is called the *visual* map.

Definition 2.1. (1) The Busemann function $\beta : \partial(\mathbb{H}^n) \times \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ is defined as follows: for $\xi \in \partial(\mathbb{H}^n)$ and $x, y \in \mathbb{H}^n$,

$$\beta_\xi(x, y) = \lim_{r\to\infty} d(x, \xi_r) - d(y, \xi_r)$$

where ξ_r is a geodesic ray tending to ξ as $r \to \infty$.

(2) For $u \in \mathrm{T}^1(\mathbb{H}^n)$, the unstable horosphere $\mathcal{H}_u^+ \subset \mathrm{T}^1(\mathbb{H}^n)$ denotes the set

$$\{v \in \mathrm{T}^1(\mathbb{H}^n) : v^- = u^-, \beta_{u^-}(\pi(u), \pi(v)) = 0\},$$

and the stable horosphere $\mathcal{H}_u^- \subset \mathrm{T}^1(\mathbb{H}^n)$ denotes the set

$$\{v \in \mathrm{T}^1(\mathbb{H}^n) : v^+ = u^+, \beta_{u^+}(\pi(u), \pi(v)) = 0\}.$$

The image under π of a horosphere $\mathcal{H}$ in $\mathrm{T}^1(\mathbb{H}^n)$ based at ξ is called a horosphere in $\mathbb{H}^n$ based at ξ, and hence is of the form $\{y \in \mathbb{H}^n : \beta_\xi(x, y) = 0\}$ for some $x \in \pi(\mathcal{H})$. Note that β is invariant by isometries, that is, for $g \in G$ and $x, y \in \mathbb{H}^n$, $\beta_\xi(x, y) = \beta_{g(\xi)}(g(x), g(y))$, and that β is an analytic function.

Let Γ be a torsion-free and non-elementary discrete subgroup of G and set $X := \Gamma\backslash\mathbb{H}^n$. Both the natural projection maps $\mathbb{H}^n \to X$ and $\mathrm{T}^1(\mathbb{H}^n) \to \mathrm{T}^1(X)$ will be denoted by $\mathbf{p}$. Denote by $\tilde{W}_u^s$ the weak stable manifold $\mathrm{Viz}^{-1}(u) = \{v \in \mathrm{T}^1(\mathbb{H}^n) : v^+ = u^+\}$ for the geodesic flow.

Definition 2.2. Consider a neighborhood P of u in $\mathcal{H}_u^+$, and a neighborhood T of u in $\mathrm{Viz}^{-1}(u)$. For each $t \in T$ and $p \in P$, the horosphere $\mathcal{H}_t^+$ intersects $\mathrm{Viz}^{-1}(u)$ at a unique vector, say, $w(t, p) \in \mathrm{T}^1(\mathbb{H}^n)$. The map $(p, t) \to w(t, p)$ provides a local chart of a neighborhood of u in $\mathrm{T}^1(\mathbb{H}^n)$.

- We call a set $B(u) = \{w(t, p) \in \mathrm{T}^1(\mathbb{H}^n) : t \in T, p \in P\}$ a box about u if for some $\epsilon_0 > 0$, the ϵ_0-neighborhood of $B(u)$ injects to $\mathrm{T}^1(X)$ under $\mathbf{p}$. We write $B(u) = T \times P$ and $tp = w(t, p)$.
- Fixing $t \in T$, $tP := \{tp : p \in P\} \subset \mathcal{H}_t^+$ is called a plaque at t and fixing $p \in P$, the image $Tp := \{tp : t \in T\} \subset \mathrm{Viz}^{-1}(p)$ is called a transversal.

The holonomy map between two transversals Tp and Tp' of B is simply given by $tp \to tp'$ for each $t \in T$.

Let $B = T \times P$ be a box. For small $\epsilon > 0$, let $T_{\epsilon+}$ be the ϵ-neighborhood of T in the same transversal, and $T_{\epsilon-} = \{t \in T : d(t, \partial T) > \epsilon\}$. Put $B_{\epsilon\pm} = T_{\epsilon\pm} \times P$.

2.2. Let $\tilde{S}$ be a complete codimension one subspace of $\mathbb{H}^n$ which is isometric to one of the following: a horosphere, a totally geodesic subspace or a hyperbolic sphere. Let $\tilde{E} \subset \mathrm{T}^1(\mathbb{H}^n)$ denote the unstable horosphere based at $\tilde{S}$ if $\tilde{S}$ is a horosphere, the full unit normal bundle to $\tilde{S}$ if $\tilde{S}$ is totally geodesic, and the outward unit normal bundle to $\tilde{S}$ if $\tilde{S}$ is a hyperbolic sphere. We set $E = \mathbf{p}(\tilde{E}) \subset \mathrm{T}^1(X)$.

Lemma 2.3. *If $g(\tilde{E}) \cap \tilde{E} \neq \emptyset$ for $g \in G$, then $g \in G_{\tilde{E}}$. Hence the canonical projection map $\Gamma_{\tilde{E}} \backslash \tilde{E} \to \mathbf{p}(E)$ is a bijection.*

Proof. Suppose $g(\tilde{E}) \cap \tilde{E} \neq \emptyset$. If $\tilde{E} = \mathcal{H}^+$, then $\tilde{E} = \mathcal{H}^+_u$ for any $u \in \tilde{E}$. Hence if $u, g(u) \in \tilde{E}$, then $\mathcal{H}^+_u = \mathcal{H}^+_{g(u)}$. On the other hand, $\mathcal{H}^+_{g(u)} = g(\mathcal{H}^+_u)$. Hence $g \in G_{\tilde{E}}$. If $\tilde{S}$ is a hyperbolic sphere of radius r, then $g^{-r}(\tilde{S})$ is a point $o \in \mathbb{H}^n$ and $g^{-r}(\tilde{E})$ is the set of all vectors in $\mathrm{T}^1(\mathbb{H}^n)$ based at o. As the geodesic flow commutes with the G-action, we may assume without loss of generality that $\tilde{S} = \{o\}$. Then the assumption implies that $g(o) = o$ and hence $g(\tilde{E}) = \tilde{E}$. If $\tilde{S}$ is a geodesic subspace of codimension one, then $G_{\tilde{E}} = G_{\tilde{S}}$, as G acts as isometries. If we define $v^\perp$ to be set of all $x \in \mathbb{H}^n$ such that the geodesic connecting x and $\pi(v)$ is orthogonal to v, then $\tilde{S} = v^\perp$ for any $v \in \tilde{E}$. Hence if $gv, v \in \tilde{E}$, then $\tilde{S} = v^\perp = (gv)^\perp$. Since $(gv)^\perp = g(v^\perp)$, we have $g \in G_{\tilde{S}}$ and hence $g \in G_{\tilde{E}}$. □

The visual map Viz restricted to $\tilde{E}$ is a diffeomorphism onto $\partial\mathbb{H}^n - \partial\tilde{S}$. For $v \in \tilde{E}$, let $\xi_v : \mathcal{H}^+_v - \mathrm{Viz}^{-1}(\partial\tilde{S}) \to \tilde{E}$ be the map given by

$$\xi_v(u) = \mathrm{Viz}^{-1}(u^+) \cap \tilde{E}.$$

Then ξ_v is a diffeomorphism onto $\tilde{E} - \{-v\}$ where $-v$ is the vector with the same base point as v but in the opposite direction. Let $q_v : \tilde{E} - \{-v\} \to \mathcal{H}^+_v$ be the map given by

$$q_v(w) = \mathrm{Viz}^{-1}(w^+) \cap \mathcal{H}^+_v.$$

Then q_v is a diffeomorphism onto $\mathcal{H}^+_v - \mathrm{Viz}^{-1}(\partial\tilde{S})$ and is the inverse of ξ_v.

Proposition 2.4. *There exist $C_1 > 0$ and $\epsilon_0 > 0$ such that*

(1) *if $v, w \in E$ and $d(v, w) < \epsilon_0$, then*
$$d(q_v(w), w) < C_1 d(w, v)^2 \quad \text{and} \quad |\beta_{w^+}(\pi(q_v(w)), \pi(w))| \leq C_1 d(w, v)^2;$$

(2) *if $v \in E$ and $u \in \mathcal{H}^+_v$ with $d(u, v) < \epsilon_0$ then*
$$d(\xi_v(u), u) < C_1 d(u, v)^2 \ \text{ and } \ |\beta_{u^+}(\pi(\xi_v(u)), \pi(u))| \leq C_1 d(u, v)^2.$$

Proof. Since ξ_v is the inverse of q_v, (2) follows from (1). Note that the claim is void if S is a horosphere, since $q_v(w) = w$ in that case. Consider the case when S is totally geodesic. By applying an appropriate isometry of $\mathbb{H}^n$, we can reduce the situation to the following: $n = 2$, $\mathbb{H}^2 = \{x + iy : y > 0\}$ is the upper half plane and $\tilde{E}$ is the set of normal vectors based on the semi-circle $\{x + iy \in \mathbb{H}^2 : x^2 + y^2 = 1\}$ and v is the upward unit normal vector at i. We

can identify $\mathrm{T}^1(\mathbb{H}^2)$ with $\mathrm{SL}_2(\mathbb{R})$ where the left translation action of $\mathrm{SL}_2(\mathbb{R})$ corresponds to the isometric action. Let

$$h(t) = \left[\begin{smallmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{smallmatrix}\right]$$

for $t \in \mathbb{R}$. Then we may assume $w = vh(t)$ for some $t \in \mathbb{R}$. Then $d(v, w) = 2t$. Now $q_v(w) \in \mathcal{H}_v^+$ and $q_v(w)^+ = w^+$. Therefore, $w \in g^{\ell_v(w)}\mathcal{H}^-_{q_v(w)}$ where $\ell_v(w) := \beta_{w^+}(\pi(w), \pi(q_v))$. There exists $n^-(y_w) = \left[\begin{smallmatrix} 1 & 0 \\ y_w & 1 \end{smallmatrix}\right]$ such that $q_v(w) = vn^-(y_w)$. And there exists $n^+(x_w) = \left[\begin{smallmatrix} 1 & x_w \\ 0 & 1 \end{smallmatrix}\right]$ such that $w = q_v(w)n^+(x_w)a(\ell_v(w))$ where $a(r) = \left[\begin{smallmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{smallmatrix}\right]$.

Then

$$d(q_v(w), w) = d(e, n^+(x_v)a(\ell_v(w))) \ll |\ell_v(w))|.$$

Hence it suffices to show $|\ell_v(w))| \ll t^2$. Since $w = vh(t) = vn^-(y_w)n^+(x_w)a(\ell_v(w))$, we have $h(t) = n^-(y_w)n^+(x_w)a(\ell_v(w))$. Now by applying both sides of this equation to $(1, 0) \in \mathbb{R}^2$ from the right, we get

$$(\cosh(t), \sinh(t)) = (e^{\ell_v(w)/2}, e^{-\ell_v(w)/2}x_w). \tag{2.1}$$

Therefore $\ell_v(w)/t^2 \to 1$ as $t \to 0$. So if we choose $\eta_0 > 0$ sufficiently small, then

$$|\beta_{w^+}(\pi(w), \pi(q_v(w))| = |\ell_v(w)| \leq 2t^2.$$

Now consider the case when S is a sphere of radius r_0. We may again assume without loss of generality that $n = 2$, $\mathbb{H}^2$ is the upper half-plane and $\tilde{E}$ is the set of outward normal vectors based at $\{x + iy \in \mathbb{H}^2 : d(x + iy, i) = r_0\}$ and v is the upward normal unit vector at $e^{r_0}i$. Then $\tilde{E} = \{vk(t) : t \in (-\pi, \pi]\}$ where $k(t) = \left[\begin{smallmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{smallmatrix}\right]$. Writing $w = vk(t)$, we deduce similarly as before that $q_v(w) = vn^-(y_w)$ and $w = vk(t) = vn^-(y_w)n^+(x_w)a(\ell_v(w))$ and hence $k(t) = n^-(y_w)n^+(x_w)a(\ell_v(w))$. This yields

$$(\cos(t), -\sin(t)) = (e^{\ell_v(w)/2}, e^{-\ell_v(w)/2}x_w). \tag{2.2}$$

Therefore $-\ell_v(w)/t^2 \to 1$ as $|t| \to 0$. Since $c_1 \cdot |t| \leq d(vk(t), v) \leq c_2 \cdot |t|$ with $c_1 > 0$ and $c_2 > 0$ being uniform constants for all small $|t|$, we conclude that for sufficiently small $\eta > 0$,

$$d(q_v(w), w) \ll |\beta_{w^+}(\pi(w), \pi(q_v(w))| = |\ell_v(w)| \leq 2t^2.$$

□

In the rest of this section, we fix a box $B = T \times P$ whose ϵ_0-neighborhood injects to $\mathrm{T}^1(X)$. Let

$$C_2 = \max\{d(t, tp) : t \in T_{\epsilon_0+}, p \in P\}.$$

Corollary 2.5. *Let $r > 0$. Let $v = g^{-r}(\gamma t) \in \tilde{E}$ and $w = \xi_v(g^{-r}(\gamma tp)) \in \tilde{E}$ for some $t, \in T, p \in P, \gamma \in \Gamma$. Then*

(1) *$d(g^{-r}(\gamma t), g^{-r}(\gamma tp) \leq C_2 e^{-r}$;*
(2) *$d(g^{-r}(\gamma tp), w) \leq C_1 C_2^2 e^{-2r}$;*
(3) *$d(v, w) \leq (C_1 C_2^2 e^{-r} + C_2)e^{-r}$.*

Proof. Since $g^{-r}(tp) \in \mathcal{H}_t^+$, (1) follows. (2) follows from (1) and Prop. 2.4. (3) follows from (1) and (2). $\square$

Lemma 2.6. (1) *For $\gamma \in \Gamma$ and any $r \in \mathbb{R}$, $\# T \cap g^r(\gamma^{-1}\tilde{E}) \leq 1$.*
(2) *If $t \in T \cap g^r(\Gamma\tilde{E})$, then $\gamma \in \Gamma$ satisfying $t \in g^r(\gamma^{-1}\tilde{E})$ is unique up to the left multiplication by $\Gamma_{\tilde{E}}$.*

Proof. Since $\mathrm{Viz}(T)$ is a singleton, so is $\mathrm{Viz}(g^{-r}(\gamma T))$. Now (1) follows since Viz restricted to $\tilde{E}$ is injective. (2) follows from Lemma 2.3. $\square$

Definition 2.7. For $r > 1$, $t \in T$ and $\gamma \in \Gamma$ with $t = g^r(\gamma^{-1}\tilde{E})$, we set

$$E_{r,t,\gamma} := \xi_{g^{-r}(\gamma t)}(g^{-r}(\gamma tP)) \subset \tilde{E}.$$

Proposition 2.8. *Let $0 < \epsilon < \epsilon_0$. Then for all sufficiently large $r \gg 1$, $t \in T$ and $\gamma \in \Gamma$ with $t \in g^r(\gamma^{-1}\tilde{E})$, we have*

$$E_{r,t,\gamma} \subset g^{-r}(\gamma B_{\epsilon+}).$$

Proof. Let $v := g^{-r}(\gamma t)$ and $w := \xi_v(g^{-r}(\gamma tp))$ for $p \in P$. We need to show that $w \in g^{-r}(\gamma B_{\epsilon+})$. Then by Corollary 2.5,

$$d(v, w) \leq (C_1 C_2^2 e^{-r} + C_2)e^{-r}.$$

Hence if r is large enough, then by Proposition 2.4,

$$d(q_w(v), v) \leq C_1 d(v, w)^2 \leq \epsilon.$$

Since $q_w(v)^+ = v^+$, if we put $t_1 = g^r(\gamma^{-1}q_w(v))$, then

$$d(t_1, t) \leq d(\gamma^{-1}q_w(v), \gamma^{-1}v) = d(q_w(v), v) < \epsilon \quad \text{for all large } r > 0.$$

Since $g^r(\gamma^{-1}w)$ and tp are on the same transversal Tp, it follows that $w = g^{-r}(\gamma t_1 p)$. Hence $w \in g^{-r}(\gamma B_{\epsilon+})$. $\square$

Proposition 2.9. *If $tp \in T_\epsilon^- \times P$ and $\gamma \in \Gamma$ satisfy $tp \in g^r(\gamma^{-1}\tilde{E})$, then for all sufficiently large $r \gg 1$, there exists $t_1 \in T \cap g^r(\gamma^{-1}\tilde{E})$ such that $g^{-r}(\gamma tp) \in E_{r,t_1,\gamma}$.*

Proof. Let $v := g^{-r}(\gamma tp) \in \tilde{E}$ and $w = \xi_v(g^{-r}(\gamma t)) \in \tilde{E}$. Since $d(w, v) \leq 2C_2 e^{-r}$, by Proposition 2.4,

$$d(w, g^{-r}(\gamma t)) \leq C_1(4C_2^2)e^{-2r}.$$

Therefore if we put $t_1 = g^r(\gamma^{-1}w)$, then $t_1^+ = t^+$, and $d(t_1, t) < 2C_2 e^{-r} < \epsilon$ for all large $r > 1$. Hence $t_1 \in T$. Since $(t_1 p)^+ = (tp)^+$, we have $v = \xi_w(g^{-r}(\gamma t_1 p)) \in E_{r,t_1,\gamma}$. $\square$

Proposition 2.10. *For $i = 1, 2$, let $t_i \in T$ and $\gamma_i \in \Gamma$ be such that $t_i \in g^r(\gamma_i^{-1}\tilde{E})$. Then for all large $r \gg 1$,*
(1) *$E_{r,t_1,\gamma_1} \cap E_{r,t_2,\gamma_2} = \emptyset$ if $\gamma_1 \neq \gamma_2$.*
(2) *$E_{r,t_1,\gamma_1} \cap E_{r,t_2,\gamma_2} = \emptyset$ if $t_1 \neq t_2$.*
(3) *$\Gamma_{\tilde{E}} E_{r,t_1,\gamma_1} = \Gamma_{\tilde{E}} E_{r,t_2,\gamma_2}$ if $\Gamma_{\tilde{E}}\gamma_1 = \Gamma_{\tilde{E}}\gamma_2$.*

Proof. By Proposition 2.8, for all large r and $\epsilon < \epsilon_0$, we have

$$E_{r,t_1,\gamma_1} \cap E_{r,t_2,\gamma_2} \subset g^{-r}(\gamma_1 B_{\epsilon+} \cap \gamma_2 B_{\epsilon+}) = \emptyset.$$

(2) follows from (1) since $t_1 \neq t_2$ implies $\gamma_1 \neq \gamma_2$ by Lemma 2.6. (3) follows since for $h \in G_{\tilde{E}}$, we have

$$\xi_{hv}(u) = h\xi_v(h^{-1}u).$$

□

By Lemma 2.6(2) and Proposition 2.10(3), the following set is well-defined (below we sometimes consider sets T, P, B, etc., as subsets of $\mathrm{T}^1(X)$; since $B_{\epsilon+}$ injects to $\mathrm{T}^1(X)$ for all small $\epsilon > 0$, this should not cause any confusion):

Definition 2.11. For $r > 1$ and $t \in T \cap g^r(E)$, set

$$E_{r,t} := \mathbf{p}(\xi_{g^{-r}(\gamma t)}(g^{-r}(\gamma t P))) \subset E$$

for any $\gamma \in \Gamma$ such that $\mathbf{p}^{-1}(t) \cap g^r(\gamma^{-1}\tilde{E}) \neq \emptyset$.

By combining Propositions 2.8, 2.9, and 2.10 we have:

Corollary 2.12. *Let $0 < \epsilon < \epsilon_0$. Then for all sufficiently large $r \gg 1$,*

$$g^{-r}(B_{\epsilon-}) \cap E \subset \bigsqcup_{t \in T \cap g^r(E)} E_{r,t} \quad \subset g^{-r}(B_{\epsilon+}) \cap E$$

where $E_{r,t_1} \cap E_{r,t_2} \neq \emptyset$ for $t_1 \neq t_2$.

2.3. Let $\{\mu_x : x \in \mathbb{H}^n\}$ be a Γ-*invariant conformal density* of dimension $\delta_\mu > 0$ on $\partial(\mathbb{H}^n)$. That is, each μ_x is a non-zero finite Borel measure on $\partial(\mathbb{H}^n)$ satisfying for any $x, y \in \mathbb{H}^n$, $\xi \in \partial(\mathbb{H}^n)$ and $\gamma \in \Gamma$,

$$\gamma_* \mu_x = \mu_{\gamma x} \quad \text{and} \quad \frac{d\mu_y}{d\mu_x}(\xi) = e^{-\delta_\mu \beta_\xi(y,x)},$$

where $\gamma_*\mu_x(F) = \mu_x(\gamma^{-1}(F))$ for any Borel subset F of $\partial(\mathbb{H}^n)$. We assume that μ_x is atom-free.

Definition 2.13. We consider the measure on $\tilde{E}$ given by

$$d\mu_{\tilde{E}}(v) = e^{\delta_\mu \beta_{v^+}(o, \pi(v))}\, d\mu_o(v^+)$$

(it is easy to check that this definition is independent of the choice of $o \in \mathbb{H}^n$).

Noting that $\gamma_* \mu_{\tilde{E}} = \mu_{\gamma(\tilde{E})}$ for any $\gamma \in \Gamma$, we let μ_E the measure on E induced by $\mu_{\tilde{E}}$.

By Corollary 2.12, we have, for any box $B = T \times P$, $\Psi \in C(B)$, and $f \in C(E)$,

$$\int_{u \in E} \Psi(g^r(u)) f(u)\, d\mu_E(u) = \sum_{t \in T \cap g^r(E)} \int_{E_{r,t}} \Psi(g^r(u)) f(u)\, d\mu_E(u). \tag{2.3}$$

The conformal density $\{\mu_x\}$ induces a Γ-invariant family of measures $\mu_{\mathcal{H}^+}$ and $\mu_{\mathbf{p}(\mathcal{H}^+)}$ on the strong unstable horospherical foliation on $\mathrm{T}^1(\mathbb{H}^n)$ and on $\mathrm{T}^1(X)$ respectively.

It can be easily checked that for any $r \in \mathbb{R}$,

$$g_*^r \mu_{\mathcal{H}_u^+} = e^{-\delta_\mu r} \mu_{\mathcal{H}^+_{g^r(u)}}. \tag{2.4}$$

Proposition 2.14. *Given $\epsilon > 0$ the following holds for all sufficiently large $r > 0$ and $t \in T \cap g^r(E)$: for any $\Psi \in C(B)$ and $f \in C(E)$,*

$$\begin{aligned}(1+\epsilon)^{-1} f_\epsilon^-(g^{-r}(t)) e^{-\delta_\mu r} \int_{p \in tP} \Psi_\epsilon^-(tp) \mu_{\mathcal{H}_t^+}(p) &\le \int_{w \in E_{r,t}} \Psi(g^r(w)) f(w) d\mu_E(w) \\ &\le (1+\epsilon) f_\epsilon^+(g^{-r}(t)) e^{-\delta_\mu r} \int_{p \in tP} \Psi_\epsilon^+(tp) \mu_{\mathcal{H}_t^+}(p)\end{aligned}$$

where $\Psi_\epsilon^\pm \in C(B_{\epsilon\pm})$ are defined by

$$\Psi_\epsilon^+(u) := \sup_{d(u,v) \le \epsilon} \Psi(v) \quad \text{and} \quad \Psi_\epsilon^-(u) = \inf_{d(u,v) \le \epsilon} \Psi(v); \tag{2.5}$$

and $f_\epsilon^\pm \in C(E)$ are given by

$$f_\epsilon^+(u) := \sup_{w \in E: d(u,w) \le \epsilon} f(w) \quad \text{and} \quad f_\epsilon^-(u) = \inf_{w \in E: d(u,w) \le \epsilon} f(w). \tag{2.6}$$

Proof. Since $d(v, w) \le \epsilon$ for $v = g^{-r}\gamma t$ and $w = \xi_v(g^{-r}(\gamma t p)) \in E_{r,t}$ for all sufficiently large r by Corollary 2.5, we have

$$f_\epsilon^-(g^{-r}(t)) \le f(w) \le f_\epsilon^+(g^{-r}(t)).$$

On the other hand, there exists $C_3 > 0$ such that for any $v = g^{-r}(\gamma t) \in E$ and $w = \xi_v(g^{-r}(\gamma t p)) \in E$ with $\gamma \in \Gamma$ and $p \in P$, we get, by Proposition 2.4 and Corollary 2.5,

$$\frac{d\mu_E(w)}{d\mu_{\mathcal{H}_v^+}(g^{-r}(\gamma t p))} \in [(1 + C_3 e^{-2r})^{-1}, 1 + C_3 e^{-2r}] \subset [(1+\epsilon)^{-1}, 1+\epsilon],$$

for all large $r \gg 1$. Since $d\mu_{\mathcal{H}_v^+}(g^{-r}(tp)) = e^{-\delta_\mu r} d\mu_{\mathcal{H}_t^+}(tp)$, the claim follows. $\square$

Corollary 2.15. *Let $\epsilon > 0$, $\Psi \in C(B)$ and $f \in C(E)$. For all large $r > 1$, we have*

$$\begin{aligned}(1+\epsilon)^{-1} \sum_{t \in T \cap g^r(E)} \psi_\epsilon^-(t) f_\epsilon^-(g^{-r}(t)) &\le e^{\delta_\mu r} \int_E \Psi(g^r(u)) f(u)\, d\mu_E(u) \\ &\le (1+\epsilon) \sum_{t \in T \cap g^r(E)} \psi_\epsilon^+(t) f_\epsilon^+(g^{-r}(t)).\end{aligned}$$

where $\psi_\epsilon^\pm(t) = \int_{tP} \Psi_\epsilon^\pm(tp)\, d\mu_{\mathcal{H}_t^+}(p)$.

Proof. The claim follows immediately from Proposition 2.14 and (2.3). $\square$

Lemma 2.16. [36, Lem 1.16] *The collection $\{\mu_{\mathcal{H}^+}\}$ of measures on the horospherical leaves $\mathbf{p}(\mathcal{H}^+)$ is a Haar system in the sense that for all boxes*

$B = T \times P$ and all continuous maps $\Psi : \mathrm{T}^1(X) \to \mathbb{R}$ with compact support included in $\mathbf{p}(B)$, the map

$$t \in T \mapsto \int_{tP} \Psi(tp)\, d\mu_{\mathcal{H}_t^+}$$

is continuous.

Definition 2.17. A box $B = T \times P$ is called admissible with respect to the family $\{\mu_{\mathcal{H}^+}\}$, if every plaque has a positive measure with respect to $\{\mu_{\mathcal{H}^+}\}$, that is, $\mu_{\mathcal{H}_t^+}(tP) > 0$ for all $t \in T$.

Corollary 2.18. *For any $u \in \mathrm{T}^1(\mathbb{H}^n)$, there exists an admissible box centered at u with respect to $\{\mu_{\mathcal{H}^+}\}$.*

Proof. Since $\mathcal{H}_u^+$ is homeomorphic to $\partial(\mathbb{H}^n) - \{u^-\}$ and $\mu_o(u^-) = 0$ by the atom-free assumption, there exists a relatively compact neighborhood $P \subset \mathcal{H}_u^+$ of u with $\mu_{\mathcal{H}_u^+} > 0$ and $\mathbf{p}$ is injective on P. By Lemma 2.16, there exists a neighborhood $T \subset \mathrm{Viz}^{-1}(u)$ of u such that $\mu_{\mathcal{H}_t^+}(tP) > 0$ for all $t \in T$. As P is relatively compact, we can take T small enough so that some neighborhood of $B(u) = T \times P$ injects to $\mathrm{T}^1(X)$. Hence $B(u) = T \times P$ is admissible. $\square$

In the following proposition and corollary, we assume that $B = T \times P$ is an admissible box with respect to $\{\mu_{\mathcal{H}^+}\}$. We let $\psi \in C(T)$ and define a function Ψ on B by

$$\Psi(tp) = \frac{\psi(t)}{\mu_{\mathcal{H}_t^+}(tP)}$$

for all $tp \in B$. By Lemma 2.16, $\Psi \in C(B)$. The following proposition will allow us to compare the transversal intersection with the mixing of the geodesic flow with respect to m^{BMS}; this will be carried out in Theorem 3.9).

Proposition 2.19. *For any given $\epsilon > 0$, the following holds for all large $r \gg 1$: for any $f \in C(E)$,*

$$(1+\epsilon)^{-1} \int_E \Psi_\epsilon^-(g^r(w)) f_\epsilon^-(w)\, d\mu_E(w) \le e^{-\delta_\mu r} \sum_{t \in T \cap g^r(E)} \psi(t) f(g^{-r}(t))$$
$$\le (1+\epsilon) \int_E \Psi_\epsilon^+(g^r(w)) f_\epsilon^+(w) d\mu_E(w).$$

where $\Psi_\epsilon^\pm$ and $f_\epsilon^\pm \in C(E)$ are given as in (2.5) *and* (2.6) *respectively.*

Proof. Since the support of Ψ_ϵ^- is contained in $B_{\epsilon-}$, by (2.12), we have

$$\int_E \Psi_\epsilon^-(g^r(w)) f_\epsilon^-(w)\, d\mu_E(w) = \sum_{t \in T \cap g^r(E)} \int_{E_{r,t}} \Psi_\epsilon^-(g^r(w)) f_\epsilon^-(w)\, d\mu_E(w). \tag{2.7}$$

If $w \in E_{r,t,\gamma}$ for some $\gamma \in \Gamma$ and is of the form $\xi_v(g^{-r}(\gamma t p))$ for $v = g^{-r}(\gamma t)$, then by Corollary 2.5,

$$d(g^r(w), \gamma t p) \le d(w, g^{-r}(\gamma t p)) \le C_1 C_2^2 e^{-2r}$$

and

$$d(w, g^{-r}(\gamma t)) \le 2C_2 e^{-r}.$$

Hence for $w \in E_{r,t}$ and for r large enough, we have

$$\Psi_\epsilon^-(g^r(w)) \le \Psi(\gamma t p) = \Psi(t) \text{ and } f_\epsilon^-(w) \le f(g^{-r}(t)).$$

Therefore, together with (2.14), this implies that

(2.8)

$$\int_{E_{r,t}} \Psi_\epsilon^-(g^r(w)) f_\epsilon^-(w)\, d\mu_E(w) \le (1+\epsilon) e^{-\delta_\mu r} d\mu_{\mathcal{H}_t^+}(tP) \Psi(t) f(g^{-r}(t))$$
$$= (1+\epsilon) e^{-\delta_\mu r} \psi(t) f(g^{-r}(t)).$$

(2.7) and (2.8) imply the first inequality. The other inequality can be proved similarly. □

Corollary 2.20. *Let $f \in C(E)$ be such that $f_\epsilon^+ \in L^1(E, \mu_E)$ for some $\epsilon > 0$. The following hold for all large $r > 1$.*

(1) *For any $\psi \in C(T)$,*
$$\sum_{t \in T \cap g^r(E)} |\psi(t) f(g^{-r}(t))| < \infty.$$
(2) *In particular, if there exists a Γ-invariant conformal density $\{\mu_x\}$ which is atom-free and $|\mu_E| < \infty$, then for any transversal T of an admissible box $B = T \times P$ with respect to $\{\mu_{\mathcal{H}^+}\}$,*
$$\#(T \cap g^r(E)) < \infty.$$
(3) *For any $\Psi \in C_c(\mathrm{T}^1(X))$,*
$$\int_{u \in E} |\Psi(g^r(u)) f(u)|\, d\mu_E(u) < \infty.$$

Proof. The claim (1) follows immediately from Proposition 2.19 since

$$\sum_{t \in T \cap g^r(E)} |\psi(t) f(g^{-r}(t))| \le (1+\epsilon) e^{\delta_\mu r} \|\Psi_\epsilon^+\|_\infty \cdot \mu_E(|f_\epsilon^+|).$$

(2) is a special case of (1). For (3), we may assume without loss of generality that $\Phi \in C(B)$ for an admissible box B with respect to $\{\mu_{\mathcal{H}^+}\}$ by Corollary 2.18. We may also assume that both Ψ and f are non-negative functions.

Defining $\psi_\epsilon^+(t) := \int_{tP} \Psi_\epsilon^+(tp)\, d\mu_{\mathcal{H}_t^+}(p)$, we have $\psi_\epsilon^+ \in C(T_{\epsilon+})$ by Lemma 2.16. Now the claim (3) follows from (1), since Proposition 2.14 and (2.3) imply that

$$\int_{u \in E} \Psi(g^r(u)) f(u)\, d\mu_E(u) \le (1+\epsilon) e^{-\delta_\mu r} \sum_{t \in T \cap g^r(E)} \psi_\epsilon^+(t) f_\epsilon^+(g^{-r}(t))$$

□

Definition 2.21. (1) The limit set $\Lambda(\Gamma)$ of Γ is the set of all accumulation points of an orbit $\Gamma(z)$ in $\overline{\mathbb{H}}^n$ for $z \in \mathbb{H}^n$. As Γ acts properly discontinuously on $\mathbb{H}^n$, $\Lambda(\Gamma)$ is contained in $\partial(\mathbb{H}^n)$.

(2) A point $\xi \in \Lambda(\Gamma)$ is a radial limit point (or a conical limit point or a point of approximation) if for some (and hence every) geodesic ray β tending to ξ and some (and hence every) point $x \in \mathbb{H}^n$, there is a sequence $\gamma_i \in \Gamma$ with $\gamma_i x \to \xi$ and $d(\gamma_i x, \beta)$ is bounded.

(3) We denote by $\Lambda_r(\Gamma)$ the set of radial limit points for Γ.

Lemma 2.22. *If Γ is Zariski dense, $\Lambda_r(\Gamma)$ is not contained in any proper subsphere in $\partial(\mathbb{H}^n)$.*

Proof. Let L denote the smallest complete totally geodesic subspace containing the convex core of $\Lambda(\Gamma)$. Then $\Gamma(L) \subset L$. If $\Lambda(\Gamma)$ is contained in a proper subsphere, then the dimension of L is at most $n-1$. Since $\Gamma \subset G_L$ and G_L is a proper algebraic subgroup of G, this yields a contradiction. Since $\Lambda(\Gamma)$ is a minimal Γ-invariant closed subset and hence $\Lambda_r(\Gamma)$ is dense in $\Lambda(\Gamma)$, this proves the claim. □

Theorem 2.23. *Suppose that Γ is Zariski dense and that there exists a Γ-invariant density $\{\mu_x : x \in \mathbb{H}^n\}$ which is atom free and $|\mu_E| < \infty$. Then the natural map $\mathbf{p} : \Gamma_{\tilde{E}} \backslash \tilde{E} \to \mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ is proper and hence $\mathbf{p}(E)$ is closed.*

Proof. If not, there exists sequences $\gamma_i \in \Gamma$ and $e_i \in \tilde{E}$ such that $\gamma_i e_i \to v \in \mathrm{T}^1(\mathbb{H}^n)$ as $i \to \infty$, and $\gamma_i \gamma_j^{-1} \notin \Gamma_{\tilde{E}}$ for all $i \neq j$. Fix $e_0 \in \tilde{E}$. Then $e_i = h_i e_0$ for some $h_i \in G_{\tilde{E}}$ and $v = g e_0$ for some $g \in G$, and $\gamma_i h_i m_i \to g$ for some $m_i \in G_{e_0}$. Since $G_{e_0} \subset G_{\tilde{E}}$ (see the proof of Lemma 2.3), we have $h_i' := h_i m_i \in G_{\tilde{E}}$. Since the image of $g\tilde{E}$ under Viz is $\partial(\mathbb{H}^n) - \partial(g\tilde{S})$, by Lemma 2.22, there exists $h_0(e_0) \in \tilde{E}$ such that $gh_0(e_0^+) \in \Lambda_r(\Gamma)$. Hence there exist $r_i \to \infty$ and $\gamma_i' \in \Gamma$ such that $g^{r_i}\gamma_i' g h_0 e_0 \to g' e_0$ for some $g' \in G$.

Let $B = T \times P$ be an admissible box centered at $g'e_0$. Let $\epsilon > 0$ be such that $g'e_0 \in B_{3\epsilon-}$. Fix $r = r_k \gg 1$ big enough and $\gamma' = \gamma_k'$ so that $g^r(\gamma' g h_0 e_0) \in B_{2\epsilon-}$ and Proposition 2.9 and Corollary 2.20 hold with respect to $\epsilon > 0$.

Since $\gamma_i h_i' \to g$, we have $g^r(\gamma'\gamma_i h_i' h_0 e_0) \to g^r(\gamma' g h_0 e_0)$ as $i \to \infty$. Therefore $g^r(\gamma'\gamma_i h_i' h_0 e_0) \in B_{\epsilon-}$ for all large $i \gg 1$. Proposition 2.9 implies the existence of $t_i \in T \cap g^r(\gamma'\gamma_i\tilde{E})$ for all large $i \gg 1$. Moreover $t_i \neq t_j$, by Lemma 2.12, because $\gamma_i \Gamma_{\tilde{E}} \neq \gamma_j \Gamma_{\tilde{E}}$ for all $i \neq j$. This shows that $\#(T \cap g^r(E)) = \infty$. By Corollary 2.20(2), this contradicts the assumption of $|\mu_E| < \infty$. □

3. Weighted equidistribution of $g_*^r \mu_E^{\mathrm{Leb}}$

3.1. Let Γ be a torsion-free and non-elementary discrete subgroup of G and set $X := \Gamma\backslash\mathbb{H}^n$. Let $g \in G$ and $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial(\mathbb{H}^n)$. Denoting by $\mathrm{Fix}(g)$ the set of fixed points of g in $\overline{\mathbb{H}}^n$, g is called elliptic if $\mathrm{Fix}(g)$ is a singleton in $\mathbb{H}^n$,

parabolic if $\mathrm{Fix}(g)$ is a singleton in $\partial(\mathbb{H}^n)$ and loxodromic if $\mathrm{Fix}(g)$ consists of two points in $\partial(\mathbb{H}^n)$. Any element of G is one of the above three.

Definition 3.1. (1) A subgroup P of G is parabolic if $\cap_{g\in P}\mathrm{Fix}(g)$ consists of a single point $\xi \in \partial(\mathbb{H}^n)$ and if P preserves set-wise every horosphere in $\mathbb{H}^n$ based at ξ.
(2) A point $\xi \in \Lambda(\Gamma)$ is a parabolic fixed point of Γ if Γ_ξ is parabolic.
(3) We denote by $\Lambda_p(\Gamma)$ the set of parabolic fixed points for Γ.

3.2. Let $\{\mu_x\}$ and $\{\mu'_x\}$ be Γ-invariant conformal densities on $\partial(\mathbb{H}^n)$ of dimension δ_μ and $\delta_{\mu'}$ respectively. Following Roblin, we define a measure $m^{\mu,\mu'}$ on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ associated to $\{\mu_x\}$ and $\{\mu'_x\}$. As, fixing $o\in\mathbb{H}^n$, the map

$$u \mapsto (u^+, u^-, \beta_{u^-}(o, \pi(u)))$$

is a homeomorphism between $\mathrm{T}^1(\mathbb{H}^n)$ with

$$(\partial(\mathbb{H}^n)\times\partial(\mathbb{H}^n) - \{(\xi,\xi) : \xi\in\partial(\mathbb{H}^n)\})\times\mathbb{R},$$

the following defines a measure on $\mathrm{T}^1(\mathbb{H}^n)$:

Definition 3.2. Set

$$d\tilde{m}^{\mu,\mu'}(u) = e^{\delta_\mu\beta_{u^+}(o,\pi(u))}\, e^{\delta_{\mu'}\beta_{u^-}(o,\pi(u))}\, d\mu_o(u^+)d\mu'_o(u^-)dt.$$

Noting that $\tilde{m}^{\mu,\mu'}$ is Γ-invariant, it induces a measure $m^{\mu,\mu'}$ on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$. This definition is independent of the choice of $o\in\mathbb{H}^n$.

Two important densities we will consider are the Patterson-Sullivan density and the G-invariant density. We denote by δ_Γ the critical exponent of Γ, that is, the abscissa of convergence of the Poincare series $\mathcal{P}_\Gamma(s) := \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma(o))}$ for $o\in\mathbb{H}^n$. As Γ is non-elementary, we have $\delta_\Gamma>0$. Generalizing the work of Patterson [31] for $n=2$, Sullivan [41] constructed a Γ-invariant conformal density $\{\nu_x : x\in\mathbb{H}^n\}$ of dimension δ_Γ supported on $\Lambda(\Gamma)$.

We denote by $\{m_x : x\in\mathbb{H}^n\}$ a G-invariant conformal density on the boundary $\partial(\mathbb{H}^n)$ of dimension $(n-1)$, unique up to homothety. In particular each m_x is invariant under the maximal compact subgroup G_x.

Definition 3.3. (1) The measure $m^{\nu,\nu}$ on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ is called the Bowen-Margulis-Sullivan measure m^{BMS}_Γ associated with $\{\nu_x\}$ ([5], [24], [42])

$$m^{\mathrm{BMS}}_\Gamma(u) = e^{\delta_\Gamma\beta_{u^+}(o,\pi(u))}\, e^{\delta_\Gamma\beta_{u^-}(o,\pi(u))}\, d\nu_o(u^+)d\nu_o(u^-)dt.$$

(2) The measure $m^{\nu,m}$ is called the Burger-Roblin measure m^{BR}_Γ associated with $\{\nu_x\}$ and $\{m_x\}$ ([6], [36]):

$$m^{\mathrm{BR}}_\Gamma(u) = e^{(n-1)\beta_{u^+}(o,\pi(u))}\, e^{\delta_\Gamma\beta_{u^-}(o,\pi(u))}\, dm_o(u^+)d\nu_o(u^-)dt.$$

We note that the support of m^{BMS}_Γ and m^{BR}_Γ are given respectively by $\{u\in\mathrm{T}^1(X) : u^+, u^-\in\Lambda(\Gamma)\}$ and $\{u\in\mathrm{T}^1(X) : u^-\in\Lambda(\Gamma)\}$. We will sometimes omit the subscript Γ in the notation of m^{BMS}_Γ and m^{BR}_Γ.

Burger [6] showed that for a convex cocompact hyperbolic surface with δ_Γ at least $1/2$, m^{BR} is a unique ergodic horocycle invariant measure which is not supported on closed horocycles. Roblin extended Burger's result in much greater generality. By identifying the space $\Omega_{\mathcal{H}}$ of all unstable horospheres with $\partial(\mathbb{H}^n) \times \mathbb{R}$ by $\mathcal{H}^+(u) \mapsto (u^-, \beta^-_u(o, \pi(u)))$, one defines the measure $d\hat{\mu}(\mathcal{H}) = d\nu_o(\xi) e^{\delta s} ds$ for $\mathcal{H} = (\xi, s)$. Then Roblin's theorem [36, Thm. 6.6] says that if $|m^{\mathrm{BMS}}_\Gamma| < \infty$, then $\hat{\mu}$ is the unique Radon Γ-invariant measure on $\Lambda_r(\Gamma) \times \mathbb{R} \subset \Omega_{\mathcal{H}}$. This important classification result is not used in our paper but it certainly explains a reason behind the theorem that the asymptotic distribution of expanding horospheres is described by m^{BR}.

The following theorem is of independent interest, though it will not be used in the rest of the paper. Defining $\phi_0 \in C(\mathbb{H}^n)$ by

$$\phi_0(x) = |\nu_x|,$$

ϕ_0 is a Γ-invariant function which is an eigenfunction of the hyperbolic Laplacian with the eigenvalue $-\delta(n-1-\delta)$.

Theorem 3.4. *The following are equivalent:*

(1) $|m^{\mathrm{BR}}_\Gamma| < \infty$*;*
(2) Γ *is a lattice in* G*;*
(3) $\phi_0 \in L^1(\Gamma\backslash\mathbb{H}^n)$.

Proof. If Γ is a lattice, then $\{\nu_x\} = \{m_x\}$ up to homothety. Hence m^{BR}_Γ is simply the Liouville measure, in particular, the projection of a G-invariant measure of $\Gamma\backslash G$ to $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$. Hence the claim follows. Suppose $|m^{\mathrm{BR}}_\Gamma| < \infty$. Since the G-action on $\mathrm{T}^1(\mathbb{H}^n)$ is transitive, we may identify $\mathrm{T}^1(\mathbb{H}^n)$ with G/M for a compact subgroup M. We lift the measure m^{BR}_Γ to $\Gamma\backslash G$ trivially, and call it m. That is, $m(f) = m^{\mathrm{BR}}_\Gamma(\int_{x\in M} f_x dx)$ where $f_x(g) = f(gx)$ and dx is the probability Haar measure on M. Denote by U the horospherical subgroup of G whose orbits in G projects to the unstable horospheres in $\mathrm{T}^1(\mathbb{H}^n)$. Then M normalizes U and any unimodular proper closed subgroup G containing U is contained in the subgroup MU. As m^{BR}_Γ is invariant $G_{\mathcal{H}^+}$ for any unstable horosphere $\mathcal{H}^+$, it follows that m is a U-invariant finite measure on $\Gamma\backslash G$. By Ratner's theorem [35], any ergodic component, say, m_0, of m is a homogeneous measure in the sense that m_0 is an H-invariant finite measures supported on $x_0 H$ for some $x_0 \in \Gamma\backslash G$ and H is a unimodular closed subgroup G containing U. If $H \neq G$, then $H \subset MU$ and $\Gamma \cap H$ is co-compact in H. It follows by a theorem of Biberbach (cf. [4, Thm.2.25]) that $\Gamma \cap U$ is co-compact in U. Hence $H = U$. Hence we may write $m = m_1 + m_2$ where m_1 is G-invariant and m_2 is supported on a union of compact U-orbits. The projection of the support of m_2 in $\mathrm{T}^1(\mathbb{H}^n)$ is a union of compact unstable horospheres based at $\Lambda_p(\Gamma)$. It follows that m_2 must be 0, since $\hat{\mu}(\Lambda_p(\Gamma) \times \mathbb{R}) = 0$. Hence m is G-invariant. Hence the finiteness of m implies that Γ is a lattice in G. This establishes the equivalence of (1) and (2). The measure m^{BR}_Γ considered as a linear functional on $C_c(\Gamma\backslash\mathbb{H}^n)$ is equal to $\phi_0 dm$ for the hyperbolic invariant measure m on $\mathbb{H}^n$ (see [19, Lem 6.7]).

Since $\phi_0 \in L^1(\Gamma\backslash\mathbb{H}^n)$ if and only if m_Γ^{BR} defines a bounded linear functional on $L^\infty(\Gamma\backslash\mathbb{H}^n)$ with the norm given by $m_\Gamma^{\mathrm{BR}}(1) = \int_{\Gamma\backslash\mathbb{H}^n}\phi_0 dm < \infty$, the equivalence of (1) and (3) follows. $\square$

3.3. Let $\tilde{S}$ and $\tilde{E}$ be as in the subsection 2.2. The following measures are special case of 2.13:

Definition 3.5. (1) Set

$$d\mu_{\tilde{E}}^{\mathrm{Leb}}(v) = e^{(n-1)\beta_{v^+}(o,\pi(v))}dm_o(v^+).$$

The measure $\mu_{\tilde{E}}^{\mathrm{Leb}}$ is G-invariant ($g_*\mu_{\tilde{E}}^{\mathrm{Leb}} = \mu_{g(\tilde{E})}^{\mathrm{Leb}}$); in particular, it is the Lebesgue measure on $\tilde{E}$ as it is preserved under the action of $G_{\tilde{E}}$.

(2) Set

$$d\mu_{\tilde{E}}^{\mathrm{PS}}(v) = e^{\delta_\Gamma\beta_{v^+}(o,\pi(v))}d\nu_o(v^+).$$

We note that $\mu_{\tilde{E}}^{\mathrm{PS}}$ is a Γ-invariant measure.

We denote by μ_E^{Leb} and μ_E^{PS} the measures on $E = \mathbf{p}(\tilde{E})$ induced by $\mu_{\tilde{E}}^{\mathrm{Leb}}$ and $\mu_{\tilde{E}}^{\mathrm{PS}}$ respectively.

In particular, we have families of measures $\mu^{\mathrm{PS}} = \{\mu_{\mathcal{H}^+}^{\mathrm{PS}}\}$ and $\mu^{\mathrm{Leb}} = \{\mu_{\mathcal{H}^+}^{\mathrm{Leb}}\}$ on the strong unstable foliation satisfying

$$\mu_{g^r(\mathcal{H}^+)}^{\mathrm{PS}}(g^r(F)) = e^{\delta_\Gamma r}\mu_{\mathcal{H}^+}^{\mathrm{PS}}(F) \quad\text{and}\quad \mu_{g^r(\mathcal{H}^+)}^{\mathrm{Leb}}(g^r(F)) = e^{\delta_\Gamma r}\mu_{\mathcal{H}^+}^{\mathrm{Leb}}(F)$$

for any Borel subset F of $\mathbf{p}(\mathcal{H}^+)$,

A transverse measure for the strong unstable foliation is a collection $\{\nu_T\}$ of Radon measures on each transversal to the foliation. We say $\{\nu_T\}$ holonomy invariant if for all holonomy maps $\eta : T \to T'$,

$$\eta_*\nu_T = \nu_{T'}.$$

Given a transverse holonomy-invariant measure ν and a Haar system α, we can define a *product* measure $\nu\circ\alpha$ on $\mathrm{T}^1(X)$ as follows: for $\psi \in C_c(\mathrm{T}^1(X))$ with support contained in a box $B = T \times P$,

$$v \circ \alpha(\psi) = \int_T \int_{p\in tP} \psi(tp)\; d\alpha(p)d\nu_T(t).$$

The holonomy invariance of ν implies that the above is well-defined, that is, independent of the choice of a transversal T. It is clear that for any pair of boxes B_1 and B_2, $v\circ\alpha|_{C_c(B_1)}$ and $v\circ\alpha|_{C_c(B_2)}$ agree on $C_c(B_1 \cap B_2)$. Therefore this definition extends uniquely on to a linear functional on $C_c(\mathrm{T}^1(X))$ using a partition of unity argument (cf. [22, Lem 4.2]).

For each transversal T of $\mathrm{T}^1(\mathbb{H}^n)$, set

$$d\tilde{\mu}_T(u) = \exp(-\delta s)d\nu_o(u^-)ds$$

where $s = \beta_{u^-}(\pi(u), o)$. Then the collection $\{\tilde{\mu}_T : T\}$ is Γ-invariant and holonomy invariant, and hence we obtain a collection $\mu = \{\mu_T\}$ of transverse

holonomy invariant measures on transversals to the strong unstable foliation. We have

$$m^{\mathrm{BMS}} = \mu \circ \mu^{\mathrm{PS}} \quad \text{and} \quad m^{\mathrm{BR}} = \mu \circ \mu^{\mathrm{Leb}}.$$

3.4. We assume that $|m_\Gamma^{\mathrm{BMS}}| < \infty$ in the rest of this section. This implies that Γ is of divergent type, that is, $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d(o,\gamma o)} = \infty$ and that the Γ-invariant conformal density of dimension $\delta = \delta_\Gamma$ is unique up to homothety (see [36, Coro.1.8]).

Hence, up to homothety, ν_x is the weak-limit as $s \to \delta_\Gamma^+$ of the family of measures

$$\nu_{x,o}(s) := \frac{1}{\sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)}} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma o)} \delta_{\gamma o}$$

for some $o \in \mathbb{H}^n$.

Since $|m_\Gamma^{\mathrm{BMS}}| < \infty$, the Γ-action on $\partial^2(\mathbb{H}^n)$ is ergodic with respect to $\nu_x \times \nu_x$ and hence the following proposition follows from [36, Pf of Thm 1.7].

Proposition 3.6. *ν_x is atom-free.*

Rudolph showed the following mixing theorem for Γ geometrically finite and Babillot showed it in general:

Theorem 3.7 (Rudolph [37], Babillot [1])**.** *For $\Psi_1, \Psi_2 \in L^2(\mathrm{T}^1(X), m^{\mathrm{BMS}})$,*

$$\lim_{r\to\infty} \int \Psi_1(x)\Psi_2(g^r(x))\, dm^{\mathrm{BMS}}(x) = \frac{1}{|m^{\mathrm{BMS}}|}\, m^{\mathrm{BMS}}(\Psi_1)\cdot m^{\mathrm{BMS}}(\Psi_2)$$

Theorem 3.8. *Let $f \in L^1(E, \mu_E^{\mathrm{PS}})$ and $\Psi \in C_c(\mathrm{T}^1(X))$.*

$$\lim_{r\to\infty} \int_{w\in E} \Psi(g^r(w)) f(w)\, d\mu_E^{\mathrm{PS}}(w) = \frac{\mu_E^{\mathrm{PS}}(f)}{|m^{\mathrm{BMS}}|} \cdot m^{\mathrm{BMS}}(\Psi). \tag{3.1}$$

Proof. In [36, Coro. 3.2], (3.1) was deduced from Theorem 3.8 for $E = \mathcal{H}_v^+$ and $f \in C_c(\mathbf{p}(\mathcal{H}_v^+))$. The general case will be deduced from this. Since $C_c(E)$ is dense in $L^1(E, \mu_E^{\mathrm{PS}})$, it is sufficient to prove (3.1) for $f \in C_c(E)$.

Fix $\epsilon > 0$ and for each $v \in E$, denote by $B_v(\epsilon)$ the ϵ-neighborhood of v in E. Suppose that the support of f is contained in $B_v(\epsilon)$. Then for any $w \in B_v(\epsilon)$, we have by Proposition 2.4, $\frac{d\mu_E(w)}{d\mu_{\mathcal{H}_v^+}(q_v(w))} \in [1 - C\epsilon, 1 + C\epsilon]$ for some $C > 1$. Therefore

$$\begin{aligned}
&\int_{w\in E} \Psi(g^r(w)) f(w)\, d\mu_E^{\mathrm{PS}}(w) \\
&\le \int_{w\in E} \Psi_\epsilon^+(g^r(q_v(w)) f(w)\, d\mu_E^{\mathrm{PS}}(w) \\
&\le (1 + C\epsilon) \int_{u\in\mathcal{H}_v^+} \Psi_\epsilon^+(g^r(u)) f(\xi_v(u))\, d\mu_{\mathcal{H}_v^+}^{\mathrm{PS}}(u).
\end{aligned}$$

Hence by applying the case of $E = \mathbf{p}(\mathcal{H}^+)$ and using $\mu_E^{\mathrm{PS}}(f) = (1 + O(\epsilon))\mu_{\mathcal{H}_v^+}^{\mathrm{PS}}(f \circ \xi_v)$ again by Proposition 2.4, we have

(3.2)
$$\limsup_{r\to+\infty} \int_{w\in E} \Psi(g^r(w))f(w)\, d\mu_E^{\mathrm{PS}}(w) = (1+O(\epsilon))\frac{\mu_E^{\mathrm{PS}}(f)}{|m^{\mathrm{BMS}}|} \cdot m^{\mathrm{BMS}}(\Psi_\epsilon^+).$$

Similarly we can deduce

$$\liminf_{r\to+\infty} \int_{w\in E} \Psi(g^r(w))f(w)\, d\mu_E^{\mathrm{PS}}(w) = (1+O(\epsilon))\frac{\mu_E^{\mathrm{PS}}(f)}{|m^{\mathrm{BMS}}|} \cdot m^{\mathrm{BMS}}(\Psi_\epsilon^-). \tag{3.3}$$

By the partition of unity argument, (3.2) and (3.3) holds for any $f \in C_c(E)$. Since $\epsilon > 0$ is arbitrary and $m^{\mathrm{BMS}}(\Psi_\epsilon^+ - \Psi_\epsilon^-) \to 0$ as $\epsilon \to 0$, this proves the claim. □

Let $B = T \times P$ be an admissible box with respect to $\{\mu_{\mathcal{H}^+}^{\mathrm{PS}}\}$ whose ϵ_0-neighborhood injects to $\mathrm{T}^1(X)$. The following is one of the crucial observations in this article:

Theorem 3.9 (Transversality Theorem)**.** *Let $f \in C(E) \cap L^1(E, \mu_E^{\mathrm{PS}})$ such that $\mu_E^{\mathrm{PS}}(f_\epsilon^+ - f_\epsilon^-) \to 0$ as $\epsilon \to 0$, and let $\psi \in C(T)$. Then*

$$\lim_{r\to\infty} e^{-\delta_\Gamma r} \sum_{t\in T\cap g^r(E)} \psi(t) f(g^{-r}(t)) = \frac{\mu_E^{\mathrm{PS}}(f)}{|m^{\mathrm{BMS}}|}\mu_T(\psi).$$

Proof. Let $\Psi(tp) = \frac{\psi(t)}{\mu_{\mathcal{H}_t^+}^{\mathrm{PS}}(tP)}$ for all $tp \in B = T \times P$. By Lemmas 2.16 and 3.6, $\Psi \in C(B)$. By Theorem 3.8, we conclude that

$$\lim_{r\to\infty} \int_E \Psi_\epsilon^\pm(g^r(v)) f_\epsilon^\pm(v)\, d\mu_E^{\mathrm{PS}}(v) = \frac{\mu_E^{\mathrm{PS}}(f_\epsilon^\pm) m^{\mathrm{BMS}}(\Psi_\epsilon^\pm)}{|m^{\mathrm{BMS}}|}. \tag{3.4}$$

Since $\mu_E^{\mathrm{PS}}(f_\epsilon^\pm - f) \to 0$ and $m^{\mathrm{BMS}}(\Psi_\epsilon^\pm) \to 0$ as $\epsilon \to 0$, by Proposition 2.19,

$$\lim_{r\to\infty} e^{-\delta_\Gamma r} \sum_{t\in T\cap g^r(E)} \psi(t) f(g^{-r}(t)) = \frac{\mu_E^{\mathrm{PS}}(f) m^{\mathrm{BMS}}(\Psi)}{|m^{\mathrm{BMS}}|}. \tag{3.5}$$

Now

$$m^{\mathrm{BMS}}(\Psi) = \int_T d\mu_T(t) \left(\int_{tP} \Psi(tp) d\mu_{\mathcal{H}_t^+(p)}^{\mathrm{PS}} \right) = \mu_T(\psi).$$

This completes the proof of the theorem. □

Theorem 3.10. *Let $f \in C(E) \cap L^1(E, \mu_E^{\mathrm{PS}})$ such that $\mu_E^{\mathrm{PS}}(f_\epsilon^+ - f_\epsilon^-) \to 0$ as $\epsilon \to 0$. Then for any $\Psi \in C_c(\mathrm{T}^1(X))$,*

$$\lim_{r\to\infty} e^{(n-1-\delta_\Gamma)r} \int_{u\in E} \Psi(g^r(u)) f(u)\, d\mu_E^{\mathrm{Leb}}(u) \sim \frac{\mu_{\mathcal{H}^+}^{\mathrm{PS}}(f)}{|m^{\mathrm{BMS}}|} m^{\mathrm{BR}}(\Psi).$$

Proof. Let B be an admissible box with respect to $\{\mu^{\mathrm{PS}}_{\mathcal{H}^+}\}$ and $\Psi \in C(B)$. Let $\psi^{\pm}_{\epsilon}(t) = \int_{tP} \Psi^{+}_{\epsilon}(tp)\ d\mu^{\mathrm{Leb}}_{\mathcal{H}^+_t}(p)$ as in Corollary 2.15. Then by Lemmas 2.16 and 3.6, $\psi^{\pm}_{\epsilon} \in C(T_{\epsilon\pm})$.

By Corollary 2.15, we have

$$(1+\epsilon)^{-1}e^{-\delta r}\sum_{t\in T\cap g^r(E)}\psi^{-}_{\epsilon}(t)f^{-}_{\epsilon}(g^{-r}(t)) \le e^{(n-1-\delta)r}\int_E \Psi(g^r(u))f(u)\ d\mu^{\mathrm{Leb}}_E(u)$$
$$\le (1+\epsilon)e^{-\delta r}\sum_{t\in T\cap g^r(E)}\psi^{+}_{\epsilon}(t)f^{+}_{\epsilon}(g^{-r}(t)).$$

Setting $\psi(t) := \int_{tP}\Psi(tp)\ d\mu^{\mathrm{Leb}}_{\mathcal{H}^+_t}(p)$, note that $\mu_T(\psi) = m^{\mathrm{BR}}(\Psi)$ as well as $\mu_T(\psi^{\pm}_{\epsilon}) = m^{\mathrm{BR}}(\Psi^{\pm}_{\epsilon})$. As Ψ is uniformly continuous, we have $m^{\mathrm{BR}}(\Psi^{+}_{\epsilon} - \Psi^{-}_{\epsilon}) \to 0$ as $\epsilon \to 0$.

We apply Theorem 3.9 and use the assumption of $\mu^{\mathrm{PS}}_E(f^{+}_{\epsilon} - f^{-}_{\epsilon}) \to 0$ to obtain that

$$\lim_r e^{(n-1-\delta)r}\int_E \Psi(g^r(u))f(u)\ d\mu^{\mathrm{Leb}}_E(u) = \frac{\mu^{\mathrm{PS}}_E(f)\mu_T(\psi)}{|m^{\mathrm{BMS}}|}.$$

Since $\mu_T(\psi) = m^{\mathrm{BR}}(\Psi)$, this proves the claim for $\Psi \in C(B)$ by (2.3). Since admissible boxes provide a basis of open sets in $\mathrm{T}^1(X)$ by Corollary 2.18, we can use a partition of unity argument to finish the proof. $\square$

The above proof was influenced by the work of Shapira [40].

4. Geometric finiteness of closed totally geodesic immersions

This section is on the geometry of closed totally geodesic immersions in neighborhoods of cusps. The results are applied in this section to obtain a criterion for compactness of the support of μ^{PS}_E, and in the next section to study finiteness of μ^{PS}_E.

4.1. Let Γ be a torsion free discrete subgroup of G. Let $\xi \in \partial(\mathbb{H}^n)$. In order to analyze the action of Γ_ξ on $\partial(\mathbb{H}^n) - \{\xi\}$, it is convenient to use the upper half space model $\mathbb{R}^n_+ = \{(x,y) : x \in \mathbb{R}^{n-1}, y > 0\}$ for $\mathbb{H}^n$, where ξ corresponds to ∞ and $\partial(\mathbb{H}^n) - \{\xi\}$ corresponds to $\partial(\mathbb{R}^n_+) = \{(x,0) : x \in \mathbb{R}^{n-1}\}$. Assume that ∞ is a parabolic fixed point for Γ. The subgroup Γ_∞ acts properly discontinuously via affine isometries on $\partial(\mathbb{H}^n) - \{\infty\} \cong \mathbb{R}^{n-1}$; at this stage we will treat $\mathbb{R}^{n-1}$ only as an affine space, and we shall be free to choose its origin 0 later. By a theorem of Biberbach [4, Thm.2.25], there exists a normal abelian subgroup Γ' of Γ_∞ which is of finite index. Moreover by [4, Prop. 2.2.6], there exists a minimal Γ_∞-invariant affine linear subspace L of positive dimension in $\mathbb{R}^{n-1}$ such that Γ_∞ acts co-compactly on L and Γ' acts as affine translations on L; which is uniquely determined up to parallel translations (which commute with the action of a subgroup of finite index in Γ_∞). We will call any such $L = L(\Gamma,\infty)$ a Γ-Biberbach subspace associated

to ∞. The rank of a parabolic fixed point ∞ is defined to be the rank of the $\mathbb{Z}$-module Γ' which is same as the dimension of L.

Definition 4.1. A parabolic fixed point $\xi \in \Lambda(\Gamma)$ is said to be *bounded* if $\Gamma_\xi \backslash (\Lambda(\Gamma) - \{\xi\})$ is compact. Denote by $\Lambda_{bp}(\Gamma)$ the set of all bounded parabolic fixed points.

Therefore if ∞ is a bounded parabolic fixed point, $\Lambda(\Gamma) - \{\infty\}$ is contained in $\{x \in \mathbb{R}^{n-1} : d_{\mathrm{Euc}}(x, L) \le r_0\}$ for some $r_0 > 0$.

4.2. In this subsection, let $\tilde{S} \subset \mathbb{H}^n$ be a totally geodesic subspace. We assume that the natural projection map $\Gamma_{\tilde{S}} \backslash \tilde{S} \to X = \Gamma \backslash \mathbb{H}^n$ is proper, or equivalently, its image $\Gamma \backslash \Gamma \tilde{S}$ is closed in X (cf. Lemma 6.9). Since $\tilde{S}$ is totally geodesic, the geometric boundary $\partial(\tilde{S})$ is the intersection of $\partial(\mathbb{H}^n)$ with the closure of $\tilde{S}$ in $\overline{\mathbb{H}^n}$.

Proposition 4.2. *Let $\infty \in \Lambda_p(\Gamma) \cap \partial\tilde{S}$. Let L be a Γ-Bieberbach subspace of $\partial\mathbb{H}^n - \{\infty\} \cong \mathbb{R}^{n-1}$ associated to ∞ and let $L_{\tilde{S}}$ be a maximal affine subspace of $\partial(\tilde{S}) - \{\infty\}$ parallel to L. Then an abelian subgroup of finite index in $\Gamma_\infty \cap \Gamma_{\tilde{S}}$ acts cocompactly on $L_{\tilde{S}}$ by translations.*

Proof. We choose $0 \in \mathbb{R}^{n-1}$ to be contained in L. The stabilizer G_∞ of ∞ in G is of the form MAN, where N is the unipotent radical of G_∞ which is abelian and acts transitively on $\mathbb{R}^{n-1}$ as translations, A is one dimensional group consisting of semisimple elements preserving the geodesic joining 0 and ∞, and M is a compact subgroup which commutes with A and acts on $\mathbb{R}^{n-1}$ by rotations fixing the origin 0. Note that $\Gamma_\infty \subset MN$.

Let $U = \{g \in N : gL = L\}$. Then U acts transitively on L by translations. Let Γ' be an abelian subgroup of Γ_∞ with finite index which acts cocompactly and properly discontinuously on L via translations. Since $0 \in L$, the connected component of the Zariski closure of Γ' is a connected abelian subgroup of the form $M_L U$, where $M_L \subset M$ and M_L acts trivially on L.

Let L_0 be a subspace of L which is parallel to $L_{\tilde{S}}$. Since $\Gamma' \backslash L$ is a compact Euclidean torus, the closure of the image of L_0 in $\Gamma' \backslash L$ equals the image of an affine subspace, say L_1, of L. Thus $\overline{\Gamma' L_0} = \Gamma' L_1$. For $i = 0, 1$, let $U_i = \{u \in U : uL_i = L_i\}$. Then U_i acts transitively on L_i, and $\Gamma' M_L U_1 = \overline{\Gamma' M_L U_0}$. Therefore the component of identity in $\overline{\Gamma' U_0}$ is of the form $M_1 U_1$, where $M_1 \subset M_L$ and $(\Gamma' \cap M_1 U_1) \backslash U_1 M_1$ is compact.

Since $\Gamma \tilde{S}$ is closed in $\mathbb{H}^n$ by the assumption, the set $\Gamma G_{\tilde{S}}$ is closed in G. Therefore $\overline{\Gamma' U_0} \subset \Gamma G_{\tilde{S}}$. And since $G_{\tilde{S}}$ contains the component of the identity in $\Gamma G_{\tilde{S}}$, we have $M_1 U_1 \subset G_{\tilde{S}}$. Hence $\partial(\tilde{S}) - \{\infty\}$ is invariant under the translation action of U_1. Hence $\partial(\tilde{S}) - \{\infty\}$ contains a subspace parallel to L_1, and by maximality of $L_{\tilde{S}}$, we have $L_0 = L_1$. In particular, $\Gamma' \cap M_1 U_1 \subset \Gamma' \cap G_{\tilde{S}}$ acts cocompactly on L_0, and hence on $L_{\tilde{S}}$, by translations.

Let L_2' be a $\Gamma' \cap G_{\tilde{S}}$-Bieberbach subspace of $\partial\tilde{S} - \{\infty\}$ associated to ∞. Then $\dim L_2' = \mathrm{rank}(\Gamma' \cap G_{\tilde{S}})$. Let L_2 be the subspace through 0

parallel to L_2'. Then $\Gamma' \cap G_{\tilde{S}}$ acts on L_2 via translations in U. Hence $L_2 \subset L$. By maximality of $L_{\tilde{S}}$, we conclude that $\dim L_2' = \dim L_{\tilde{S}}$. Hence $\operatorname{rank}(\Gamma' \cap M_1 U_1) = \operatorname{rank}(G' \cap G_{\tilde{S}})$. □

As a direct consequence of Proposition 4.2, we obtain the following:

Lemma 4.3 (Corank Lemma)**.** *For $\infty \in \Lambda_p(\Gamma) \cap \partial(\tilde{S})$, the co-rank of $\Gamma_\infty \cap \Gamma_{\tilde{S}}$ in Γ_∞ is at most the co-dimension of $\tilde{S}$ in $\mathbb{H}^n$. That is, for a free abelian subgroup Γ' of Γ_∞ of finite index, the rank of the $\mathbb{Z}$-module $\Gamma'/\Gamma' \cap \Gamma_{\tilde{S}}$ is at most $n - dim(\tilde{S})$.*

Proposition 4.4. *Let $\infty \in \Lambda_{bp}(\Gamma) \cap \partial(\tilde{S})$. Then*

$$\begin{cases} \infty \in \Lambda_{bp}(\Gamma_{\tilde{S}}) & \text{if } \Gamma_\infty \cap \Gamma_{\tilde{S}} \neq \{e\}; \\ \infty \notin \Lambda(\Gamma_{\tilde{S}}) & \text{if } \Gamma_\infty \cap \Gamma_{\tilde{S}} = \{e\}. \end{cases}$$

Proof. Let $L = L(\Gamma, \infty)$ and Γ' be a subgroup of finite index in Γ_∞ which acts by translations on $\mathbb{R}^{n-1}$. Then $\Lambda(\Gamma) - \{\infty\}$ is contained in a bounded neighborhood of L and hence $(\Lambda(\Gamma) - \{\infty\}) \cap \partial(\tilde{S})$ is contained in a bounded neighborhood of L intersected with $\partial(\tilde{S}) - \{\infty\}$. Denoting by $L_{\tilde{S}}$ a maximal affine subspace of $\partial(\tilde{S}) - \{\infty\}$ parallel to L, it follows that $(\Lambda(\Gamma) - \{\infty\}) \cap \partial(\tilde{S})$ is contained in a bounded neighborhood of $L_{\tilde{S}}$ in $\partial(\tilde{S}) - \{\infty\}$. If $\Gamma_\infty \cap \Gamma_{\tilde{S}}$ is non-trivial (and hence infinite), then $L_{\tilde{S}}$ is a $\Gamma_{\tilde{S}}$-Biberbach space for ∞ by Proposition 4.2 and hence $\infty \in \Lambda_{bp}(\Gamma_{\tilde{S}})$.

If $\Gamma_\infty \cap \Gamma_{\tilde{S}}$ is trivial, $\Lambda_{\Gamma_{\tilde{S}}} - \{\infty\}$ is contained in a bounded subset of $\partial(\tilde{S}) - \{\infty\}$. Hence $\infty \in \partial(\tilde{S})$ is isolated from $\Lambda(\Gamma_{\tilde{S}}) - \{\infty\}$. Since the limit set of a non-elementary hyperbolic group is perfect [12], it follows that $\Gamma_{\tilde{S}}$ is elementary, and hence $\Gamma_{\tilde{S}}$ is either parabolic or loxodromic. In the former case, $\Lambda(\Gamma_{\tilde{S}}) = \{\infty\} = \Lambda_p(\Gamma_{\tilde{S}})$, contradiction the assumption that $\Gamma_\infty \cap \Gamma_{\tilde{S}} = \{e\}$. In the latter case, $\Lambda(\Gamma_{\tilde{S}}) = \Lambda_r(\Gamma_{\tilde{S}})$ and hence ∞ is a radial limit point for $\Gamma_{\tilde{S}}$ and hence for Γ. This contradicts the assumption that $\infty \in \Lambda_p(\Gamma)$. □

Lemma 4.5. *We have*

$$\Lambda_r(\Gamma) \cap \partial(\tilde{S}) = \Lambda_r(\Gamma_{\tilde{S}}).$$

Proof. Let $\xi \in \Lambda_r(\Gamma) \cap \partial(\tilde{S})$. As $\tilde{S}$ is totally geodesic, there exists a geodesic ray, say, β, lying in $\tilde{S}$ pointing toward ξ. Since ξ is a radial limit point, $\Gamma\beta$ accumulates on a compact subset of $\mathbb{H}^n$. By the assumption that the natural projection map $\Gamma_{\tilde{S}} \backslash \tilde{S} \to X$ is proper, $\Gamma_{\tilde{S}}\beta$ accumulates on a compact subset of $\tilde{S}$. This implies $\xi \in \Lambda_r(\Gamma_{\tilde{S}})$. The other direction for the inclusion is clear. □

Bowditch established many equivalent definitions of a geometrically finite hyperbolic group in [4]. In particular, we have:

Theorem 4.6 ([2],[25], [4])**.** *Γ is geometrically finite if and only if $\Lambda(\Gamma) = \Lambda_r(\Gamma) \cup \Lambda_{bp}(\Gamma)$.*

Hence, for Γ geometrically finite, $\Lambda_p(\Gamma) = \Lambda_{bp}(\Gamma)$.

Theorem 4.7. *If Γ is geometrically finite, then $\Gamma_{\tilde{S}}$ is geometrically finite.*

Proof. Since $\Lambda(\Gamma) = \Lambda_r(\Gamma) \cup \Lambda_{bp}(\Gamma)$, it follows from Proposition 4.4 and Lemma 4.5 that $\Lambda(\Gamma_{\tilde{S}}) = \Lambda_{bp}(\Gamma_{\tilde{S}}) \cup \Lambda_r(\Gamma_{\tilde{S}})$, proving the claim by Theorem 4.6. $\square$

Definition 4.8. We say $\xi \in \Lambda_p(\Gamma) \cap \partial(\tilde{S})$ is Γ-*internal* if $[\Gamma_\xi : \Gamma_\xi \cap \Gamma_{\tilde{S}}] < \infty$.

The following is immediate from Proposition 4.2:

Corollary 4.9. *Let $\infty \in \Lambda_{bp}(\Gamma) \cap \partial(\tilde{S})$. If a Γ-Biberbach subspace of ∞ is parallel to $\partial(\tilde{S}) - \{\infty\}$, then ∞ is Γ-internal.*

Let $\tilde{E} \subset \mathrm{T}^1(\mathbb{H}^n)$ denote the set of all normal vectors to $\tilde{S}$ and $E = \mathbf{p}(\tilde{E})$. We observe that $\mathrm{supp}(\mu_E^{\mathrm{PS}}) = \{v \in \tilde{E} : v^+ \in \Lambda(\Gamma)\}$.

Proposition 4.10. *Let $\mathcal{D} \subset \tilde{S}$ be a Dirichlet domain for $\Gamma_{\tilde{S}}$, i.e., $\mathcal{D} = \{s \in \tilde{S} : d(s, a) \le d(s, \Gamma_{\tilde{S}} a)\}$ for some $a \in \tilde{S}$.*

1. *If Γ is geometrically finite, then $\Lambda(\Gamma) \cap \partial(\mathcal{D}) = \Lambda_p(\Gamma) \cap \partial(\mathcal{D})$.*
2. *If $\xi \in \Lambda_{bp}(\Gamma) \cap \partial(\mathcal{D})$ is Γ-internal, then there exists a neighborhood U of ξ in $\partial(\mathbb{H}^n)$ such that*
$$\{v \in \tilde{E} : \pi(v) \in \mathcal{D},\ v^+ \in U \cap \Lambda(\Gamma)\} = \emptyset.$$

Proof. Suppose $\xi \in \Lambda(\Gamma) \cap \partial(\mathcal{D})$. As $\overline{\mathcal{D}}$ is convex and $\Gamma_{\tilde{S}}(\overline{\mathcal{D}})$ is locally finite in $\tilde{S}$ [4, Lem. 3.5.11], ξ cannot be a radial limit point for $\Gamma_{\tilde{S}}$. By Lemma 4.5, this proves (1).

To prove (2), suppose not. Then there exists a sequence $v_m \in \tilde{E}$ with $\pi(v_m) \in \mathcal{D}$, $v_m^+ \in \Lambda(\Gamma)$ and $v_m^+ \to \xi$. Without loss of generality, we may assume $\xi = \infty$ and hence $\tilde{S}$ is a vertical plane in the upper half space model of $\mathbb{H}^n$. Note that $v_m^+ \to \infty$ implies that $\pi(v_m) \to \infty$.

Since ∞ is bounded, all $v_m^+ \in \Lambda(\Gamma) - \{\infty\}$ are contained in a bounded neighborhood of L. It follows that there exists $r > 0$ such that for all m,
$$\pi(v_m) \in B(L, r) := \{x \in \tilde{S} : d_{\mathrm{Euc}}(x, L) \le r\}.$$

As ∞ is Γ-internal, $\Gamma_\infty \cap \Gamma_{\tilde{S}}$ has finite index in Γ_∞ and hence acts as translations on $L = L(\Gamma, \infty)$ cocompactly. Hence $\Gamma_\infty \cap \Gamma_{\tilde{S}}$ acts cocompactly on $B(L, r)$ as well. This is a contradiction, since $\pi(v_m) \in \mathcal{D}$ and $\pi(v_m) \to \infty$. $\square$

Theorem 4.11. *Let Γ be geometrically finite. If every point of $\Lambda_p(\Gamma) \cap \partial(\tilde{S})$ is Γ-internal, then $\mathrm{supp}(\mu_E^{\mathrm{PS}})$ is compact.*

Proof. Let $\mathcal{D} \subset \tilde{S}$ be a Dirichlet fundamental domain for $\Gamma_{\tilde{S}}$. If the claim does not hold, there exists an unbounded sequence $v_m \in \tilde{E}$ with $\pi(v_m) \in \mathcal{D}$ and $v_m^+ \in \Lambda(\Gamma)$. Since $\Lambda(\Gamma)$ is compact, by passing to a subsequence, we have assume that $v_m^+ \to \xi$ for some $\xi \in \Lambda(\Gamma)$. By Proposition 4.10 (1) and the assumption on the geometric finiteness of Γ, $\xi \in \Lambda_{bp}(\Gamma) \cap \partial(\mathcal{D})$. Since

ξ is Γ-internal by the assumption, this yields a contradiction to Proposition 4.10 (2). $\square$

4.3. **Compactness of** $\operatorname{supp}(\mu_E^{\mathrm{PS}})$ **for Horospherical** E.

Lemma 4.12. *Let $\mathcal{H}$ be a horosphere in $\mathrm{T}^1(\mathbb{H}^n)$ based at $\xi \in \Lambda_p(\Gamma)$. Then*

$$\Gamma_{\mathcal{H}} = \Gamma_\xi.$$

Proof. If $u, \gamma(u) \in \mathcal{H}$, $\xi = u^- = \gamma(u)^- = \gamma(u^-)$. Hence $\Gamma_{\mathcal{H}} \subset \Gamma_\xi$. Suppose that $\gamma \in \Gamma_\xi$ does not preserve $\mathcal{H}$. Then such γ must be a loxodromic element. On the other hand, since ξ is a parabolic fixed point, there exists a non-identity $\gamma_0 \in \Gamma_{\mathcal{H}}$. By the well known fact that a loxodromic element and a parabolic element fixing the same element of $\partial(\mathbb{H}^n)$ cannot generate a discrete subgroup, this is a contradiction. $\square$

Theorem 4.13 (Dal'bo [7])**.** *Let Γ be geometrically finite. For a horosphere $\mathcal{H}$ in $\mathrm{T}^1(\mathbb{H}^n)$ based at $\xi \in \partial(\mathbb{H}^n)$, $E := \mathbf{p}(\mathcal{H})$ is closed in $\mathrm{T}^1(X)$ if and only if either $\xi \notin \Lambda(\Gamma)$ or $\xi \in \Lambda_{bp}(\Gamma)$.*

Theorem 4.14. *Let Γ be geometrically finite. If $E := \mathbf{p}(\mathcal{H})$ is a closed horosphere in $\mathrm{T}^1(X)$, then $\operatorname{supp}(\mu_E^{\mathrm{PS}})$ is compact.*

Proof. Let $\xi \in \partial(\mathbb{H}^n)$ be the base point for $\mathcal{H}$. The restriction of the visual map $\mathrm{Vis} : v \mapsto v^+$ induces a homeomorphism $\psi : \mathcal{H} \to \partial(\mathbb{H}^n) - \{\xi\}$. As E is closed, by Theorem 4.13, either $\xi \notin \Lambda(\Gamma)$ or ξ is bounded parabolic fixed point. If $\xi \notin \Lambda(\Gamma)$, $\Gamma_\xi = \{e\}$ and $\Lambda(\Gamma)$ is a compact subset of $\partial(\mathbb{H}^n) - \{\xi\}$. Since $\operatorname{supp}(\mu_E^{\mathrm{PS}}) = \mathbf{p}(\psi^{-1}(\Lambda(\Gamma)))$, it follows that $\operatorname{supp}(\mu_E^{\mathrm{PS}})$ is compact.

Suppose now that ξ is a bounded parabolic fixed point. By Def. 3.1 and Lemma 4.12, $\Gamma_{\mathcal{H}} \backslash (\Lambda(\Gamma) - \{\xi\})$ is compact. Since ψ induces a homeomorphism between $\Gamma_{\mathcal{H}} \backslash \mathcal{H}$ and $\Gamma_{\mathcal{H}} \backslash (\partial(\mathbb{H}^n) - \{\xi\})$, it follows that $\Gamma_{\mathcal{H}} \backslash \psi^{-1}(\Lambda(\Gamma) - \{\xi\})$ is compact and is equal to $\operatorname{supp}(\mu_E^{\mathrm{PS}})$. $\square$

5. Criterion for finiteness of μ_E^{PS}

Note that (see [8, Prop. 2]):

Lemma 5.1. *If there exists $\xi \in \Lambda_p(\Gamma)$ of rank r, then $\delta_\Gamma > r/2$.*

Let $\tilde{S}$ be a complete totally geodesic submanifold of $\mathbb{H}^n$ such that $\Gamma\tilde{S}$ is closed. Let $\tilde{E} \subset \mathrm{T}^1(\mathbb{H}^n)$ be the unit normal bundle on $\tilde{S}$. Let $E = \mathbf{p}(\tilde{E}) \subset \Gamma \backslash \mathrm{T}^1(\mathbb{H}^n)$.

Theorem 5.2. *Let Γ be geometrically finite (and torsion free). Assume that co-dimension of $\tilde{S}$ in $\mathbb{H}^n$ is 1.*

1. *If $\delta_\Gamma > 1$, then $|\mu_E^{\mathrm{PS}}| < \infty$.*
2. *If $\delta_\Gamma \le 1$, then $|\mu_E^{\mathrm{PS}}| = \infty$ if and only if there exists $\xi \in \Lambda_p(\Gamma) \cap \partial(\tilde{S})$ (of rank one) such that $\Gamma_\xi \cap \Gamma_{\tilde{S}}$ is trivial.*

Proof. We will understand μ_E^{PS} as a Borel measure on $\{v \in \tilde{E} : \pi(v) \in \mathcal{D}\}$ for a fixed Dirichlet fundamental domain $\mathcal{D}$ of $\Gamma_{\tilde{S}}$ in $\tilde{S}$. Note that if $\mathcal{O}$ is a neighborhood of $\Lambda(\Gamma)\cap\partial(\mathcal{D})$ in $\partial(\mathbb{H}^n)$, then $\{v \in \tilde{E} : \pi(v) \in \mathcal{D},\ v^+ \in \Lambda - \mathcal{O}\}$ is relatively compact in $\tilde{E}$. For neighborhood U of ξ in $\partial\mathbb{H}^n$, set

$$\mathcal{D}_U = \{v \in \tilde{E} : \pi(v) \in \mathcal{D},\ v^+ \in U \cap \Lambda(\Gamma)\}. \tag{5.1}$$

By the compactness of $\Lambda(\Gamma) \cap \partial(\mathcal{D})$, to prove finiteness of μ_E^{PS}, it suffices to show that for every $\xi \in \Lambda(\Gamma) \cap \partial(\mathcal{D})$, there exists a neighborhood U of ξ in $\partial(\mathbb{H}^n)$ such that

$$\mu_{\tilde{E}}^{\mathrm{PS}}(\mathcal{D}_U) < \infty. \tag{5.2}$$

If $\xi \in \Lambda_{bp}(\Gamma) \cap \partial(\mathcal{D})$ is Γ-internal, we have $\mathcal{D}_U = \emptyset$ for sufficiently small neighborhood U of ξ by Proposition 4.10(2), and (5.2) holds trivially.

Now we assume that $\xi \in \Lambda_{bp}(\Gamma) \cap \partial(\mathcal{D})$ is not Γ-internal. By Lemma 4.3, the co-rank of $\Gamma_\xi \cap \Gamma_{\tilde{S}}$ in Γ_ξ is one. We consider the upper half space model for $\mathbb{H}^n$ with $\xi = \infty$ as in Section 4. Setting L to be a Γ-Biberbach space of ∞, $L_0 := L \cap \partial(\tilde{S}) \neq \emptyset$ is a $\Gamma_{\tilde{S}}$-Biberbach space for ∞ which has co-dimension one in L. Without loss of generality we may assume L_0 contains the origin; note that $\dim L_0$ may possibly be 0.

Let $\gamma_0 \in \Gamma_\infty$ be such that γ_0 and $\Gamma_{\tilde{S}} \cap \Gamma_\infty$ generate a subgroup of Γ_∞ of finite index. We can write $\gamma_0 = u_w\sigma$ where $u_w \in G_\infty$ acts as the translation by $w \in \mathbb{R}^{n-1} - \partial(\tilde{S})$ and σ is a rotation on $L^\perp$ fixing L pointwise. Note that $L = L_0 + \mathbb{R}w$ and $\gamma_0^k v = v + kw$ for all $v \in L$ and $k \in \mathbb{Z}$.

Since $\infty \in \Lambda_{bp}(\Gamma)$, there exists a ball B about 0 in $L^\perp$ such that $\Lambda(\Gamma) - \{\infty\} \subset B \times L$. Since $L = L_0 + \mathbb{R}w$ and $\Gamma_{\tilde{S}} \cap \Gamma_\infty$ acts on L_0 cocompactly as translations, there is a bounded open subset F_1 in L_0 such that

$$\{v^+ \in \Lambda(\Gamma) : v \in \tilde{E},\ \pi(v) \in \mathcal{D}\} \subset B \times (F_1 + \mathbb{R}w) = \cup_{k\in\mathbb{Z}}(B \times (F_1 + \gamma_0^k F_2)),$$

where $F_2 = [0,1]w$. We have

$$\xi \in B \times (F_1 + \gamma_0^k F_2) \Rightarrow \|\xi - kw\| < \operatorname{diam}(B \times (F_1 + F_2)).$$

Since w is transversal to $\partial(\tilde{S}) - \{\infty\}$, there exists N large enough such that $\cup_{|k|\geq N}(B \times (F_1 + \gamma_0^k(F_2)))$ is bounded away from $\partial(\tilde{S}) - \{\infty\}$. Therefore there exists a neighborhood U of ∞ in $\partial(\mathbb{H}^n)$ such that

$$\mathcal{D}_U = \{v^+ \in \Lambda(\Gamma) \cap U : v \in \tilde{E},\ \pi(v) \in \mathcal{D}\} \subset B \times (F_1 + (\cup_{|k|\geq N}\gamma_0^k F_2))).$$

Fix a base point $o \in \mathbb{H}^n$ on the geodesic joining 0 and ∞. Let $\mathbf{v} : \mathbb{R}^{n-1} - \partial(\tilde{S}) \to \tilde{E}$ be the inverse of the visual map restricted to $\tilde{E}$. Then

$$\mu_{\tilde{E}}^{\mathrm{PS}}(\mathcal{D}_U) \leq \sum_{|k|\geq N} \int_{\xi \in B\times(F_1+\gamma_0^k F_2)} e^{\delta\beta_\xi(o,\pi(\mathbf{v}(\xi)))}\, d\mu_o(\xi). \tag{5.3}$$

Since $\cup_{|k|\geq N}(B \times (F_1 + \gamma_0^k(F_2)))$ is away from a neighborhood of $\partial(\tilde{S})$, the map $\pi\circ\mathbf{v}$ is uniformly continuous on $\cup_{|k|\geq N} B\times(F_1+\gamma_0^k(F_2))$, with respect to the Euclidean metric on $\mathbb{R}^{n-1}$ and the hyperbolic metric on $\mathbb{H}^n$. Hence there exists d_0 independent of all $|k| \geq N$ such that $d(\pi(\mathbf{v}(\xi)), \pi(\mathbf{v}(kw))) \leq d_0$ for

$\xi \in B\times(F_1+\gamma_0^k(F_2))$. Set $\tilde{w} := \pi(\mathbf{v}(w)) \in \tilde{S}$. Since $\tilde{S}$ is a vertical hyperplane containing 0, $\pi(\mathbf{v}(kw)) = |k|\tilde{w}$. Write $w = w_1' + w_1$ with $w_1' \in \partial(\tilde{S}) - \{\infty\}$ and w_1 is in the orthogonal complement of $\partial(\tilde{S}) - \{\infty\}$. Note that $w_1 \neq 0$. Let $\tilde{w}_1 = \pi(\mathbf{v}(w_1))$. Since $\mathbb{Z}^+\tilde{w}$ and $\mathbb{Z}^+\tilde{w}_1$ are on two rays in $\mathbb{R}^n_+$ emanating from $0 \in \partial(\mathbb{H}^n) - \{\infty\}$, there exists $d_1 > 0$ independent of k such that $d(|k|\tilde{w}, |k|\tilde{w}_1) < d_1$. By the uniform continuity of the Busemann function $\beta_\xi(x,y)$ in $\xi \in \partial\mathbb{H}^n$ for fixed $x, y \in \mathbb{H}^n$, and since $|\beta_\xi(x,y)| \le d(x,y)$, there exists $c_0 > 0$ independent of $|k| > N$ such that

(5.4)

$$|\beta_\xi(o, \pi(\mathbf{v}(\xi))) - \beta_{kw_1}(o, k\tilde{w}_1)| \le c_0 + |\beta_{kw}(o, \pi(\mathbf{v}(\xi)) - \beta_{kw_1}(o, k\tilde{w}_1)|$$

$$= c_0 + |\beta_{kw}(\pi(\mathbf{v}(\xi), k\tilde{w}_1)| \le c_0 + d_0 + d_1, \tag{5.5}$$

for all $\xi \in B \times (F_1 + \gamma_0^k(F_2))$.

On the other hand, the 2 dimensional plane in $\mathbb{R}^n_+$ containing 0, w_1 and $\tilde{w}_1$ is a vertical plane containing o and and hence a copy of H^2 which is orthogonal to $\tilde{S}$ intersecting in the geodesic joining 0 and ∞. Identifying this plane with $\mathbb{H}^2 = \{x + yi : y > 0\}$, we may write $o = i$, $w_1 = x_0$ and $\tilde{w}_1 = x_0 i$. Using a standard hyperbolic distance formula, we compute that $e^{\beta_{kw_1}(o, k\tilde{w}_1)} \asymp |k|$ where the implied constant is independent of $|k| > N$ (here $\asymp$ means that the ratio of the two functions in ξ is bounded both from below and above).

Therefore by (5.4) for any $|k| \ge N$ and for any $\xi \in B \times (F_1 + \gamma_0^k(F_2))$,

$$e^{\beta_\xi(o, \pi(\mathbf{v}(\xi)))} \asymp |k|. \tag{5.6}$$

On the other hand, using that $e^{\beta_0(o, \gamma_0^k(o))} \asymp k^2$ for all $|k| > 1$,

$$\begin{aligned} \mu_o(B \times (F_1 + \gamma_0^k F_2)) &\asymp \mu_o(\gamma_0^k(B \times (F_1 + F_2))) \\ &\asymp \int_{\xi \in B \times (F_1+F_2)} e^{\delta\beta_\xi(\gamma_0^k(o), o)} d\nu_o(\xi) \\ &\asymp e^{\delta\beta_0(\gamma_0^k(o), o)} \mu_o(B \times (F_1 + F_2)) \\ &\asymp \frac{1}{k^{2\delta}} \mu_o(B \times (F_1 + F_2)) \end{aligned} \tag{5.7}$$

where the implied constant is independent of $|k| > 1$.

Therefore by (5.6) and (5.7), we obtain

$$\sum_{|k|\ge N} \int_{\xi \in B\times(F_1+\gamma_0^k F_2)} e^{\delta\beta_\xi(o, \pi(\mathbf{v}(\xi))} \, d\mu_o(\xi) \asymp \left(\sum_{|k|\ge N} |k|^{-\delta}\right) \cdot \mu_o(B \times (F_1 + F_2)).$$

Hence if $\delta > 1$, $\sum_{|k|\ge N} |k|^{-\delta} < \infty$ and this proves (1).

Now suppose $\delta \le 1$. By Lemma 5.1, any $\xi \in \Lambda_{bp}(\Gamma) \cap \partial(\tilde{S})$ is of rank one. Hence if $\Gamma_\xi \cap \Gamma_{\tilde{S}}$ is non-trivial for every such ξ, and hence ξ is internal, Theorem 4.11 implies μ_E^{PS} is compactly supported and hence finite. Now

suppose there exists $\xi \in \Lambda_{bp}(\Gamma) \cap \partial(\tilde{S})$ such that $\Gamma_\xi \cap \Gamma_{\tilde{S}}$ is trivial. By Proposition 4.4, it follows that $\xi \notin \Lambda(\Gamma_{\tilde{S}})$. Since $\Gamma_{\tilde{S}}$ acts on $\tilde{S} \cup (\partial(\tilde{S}) - \Lambda(\Gamma_{\tilde{S}}))$ properly discontinuously, there exists a neighborhood W of ξ in $\tilde{S} \cup (\partial(\tilde{S}) - \Lambda(\Gamma_{\tilde{S}}))$ such that $W \cap \tilde{S}$ is contained in a single Dirichlet domain $\mathcal{D}$ of $\Gamma_{\tilde{S}}$.

As before we may assume $\xi = \infty$. Then $L = \mathbb{R}w$ and F_1 is trivial. Let $N > 1$, U and B be as before. Then setting $\Omega := \{v^+ \in \Lambda(\Gamma) \cap U : v \in \tilde{E}, \pi(v) \in \mathcal{D}\}$, $\Omega \subset B \times (\cup_{|k| \geq N} \gamma_0^k F_2)$. By replacing U by a smaller neighborhood of ∞ if necessary, we may assume $\pi(\mathbf{v}(U)) \subset W \cap \tilde{S}$. Since $W \cap \tilde{S} \subset \mathcal{D}$, it follows that

$$\Omega = \{v^+ \in \Lambda(\Gamma) \cap U : v \in \tilde{E}\}.$$

Take any small neighborhood $\mathcal{O}$ of Ω in $\partial(\mathbb{H}^n) - \{\infty\}$ so that $\pi(\mathbf{v}(\mathcal{O} - \partial(\tilde{S}))) \subset W$. Since $\mathcal{O} \cup \{\infty\}$ is a neighborhood of ∞, and hence contains the exterior of a large ball in $\mathbb{R}^{n-1} = \partial(\mathbb{H}^n) - \{\infty\}$, we have

$$B \times (\cup_{|k| \geq N_0} \gamma_0^k F_2) \subset \mathcal{O}$$

for some large $N_0 > N$. Therefore the above computation shows that

$$\mu_E^{\mathrm{PS}}(\mathcal{O}) \geq \mu_E^{\mathrm{PS}}(B \times (\cup_{|k| \geq N_0} \gamma_0^k F_2)) \asymp \sum_{|k| \geq N_0} |k|^{-\delta} \cdot \nu_o(B \times (F_2)).$$

Note that $B \times (\cup_{|k| \geq N_0} \gamma_0^k F_2) \cup \{\infty\}$ is a neighborhood of $\infty \in \Lambda(\Gamma)$ and that ν_o is atom-free. Hence $\nu_o(B \times (\cup_{|k| \geq N_0} \gamma_0^k F_2)) > 0$. Since $\nu_o(B \times \gamma_0^k F_2) \asymp \frac{1}{k^{2\delta}} \nu_o(B \times F_2)$, we have $\nu_o(B \times F_2) > 0$.

Therefore if $\delta \leq 1$ and hence $\sum_{|k| \geq N_0} |k|^{-\delta} = \infty$, it follows that $\mu_E^{\mathrm{PS}}(\mathcal{O}) = \infty$ for any sufficiently small neighborhood $\mathcal{O}$ of Ω. Hence $\mu_E^{\mathrm{PS}}(\Omega) = \infty$ proving (2). $\square$

Corollary 5.3. *If* $|\mu_E^{\mathrm{Leb}}| < \infty$ *then* $|\mu_E^{\mathrm{PS}}| < \infty$.

Proof. If $\delta_\Gamma > 1$ then the conclusion follows from Theorem 5.2. If every $\xi \in \partial\tilde{S} \cap \Lambda_{bp}(\Gamma)$ is internal, then by Theorem 4.11 $\mathrm{supp}(\mu_E^{\mathrm{PS}})$ is compact and the conclusion follows.

Now suppose that $\delta_\Gamma \leq 1$ and there exists $\xi \in \partial(\tilde{S}) \cap \Lambda_p(\Gamma)$ not fixed by any non-trivial element of $\Gamma_{\tilde{S}}$. Then the second part of the proof of Theorem 5.2 shows that a Dirichlet domain contains a neighborhood of ξ in $\tilde{S}$, and hence $|\mu_E^{\mathrm{Leb}}| = \infty$. $\square$

The proof of Theorem 5.2 can be adapted in a straightforward manner to obtain the following general result.

Theorem 5.4. *Let* Γ *be geometrically finite and suppose that the codimension of* $\tilde{S}$ *in* $\mathbb{H}^n$ *is* k*. If* $\delta_\Gamma > k$ *then* $|\mu_E^{\mathrm{PS}}| < \infty$. $\square$

Proposition 5.5. *If* $[\Gamma : \Gamma_{\tilde{S}}] = \infty$*, then* $\Lambda(\Gamma) \not\subset \partial_\infty(\tilde{S})$ *and hence* $|\mu_E^{\mathrm{PS}}| > 0$.

Proof. Suppose on the contrary that $\Lambda(\Gamma) \subset \partial_\infty(\tilde{S})$. Let L be the smallest complete totally geodesic subspace of $\mathbb{H}^n$ containing the convex core of $\Lambda(\Gamma)$. It follows that $\Gamma(L) \subset L$. Also by our assumption $\Lambda(\Gamma) \subset \partial(\tilde{S})$, we have $L \subset \tilde{S}$. Since $L \subset \tilde{S}$ is closed and $\Gamma_{\tilde{S}}(L) = L$, it follows that the natural inclusion map $\Gamma_{\tilde{S}}\backslash L \to \Gamma_{\tilde{S}}\backslash\tilde{S}$ is proper. Since the projection $\Gamma_{\tilde{S}}\backslash\tilde{S} \to \Gamma\backslash\mathbb{H}^n$ is proper, we have that the composition map $I : \Gamma_{\tilde{S}}\backslash L \to \Gamma\backslash\mathbb{H}^n$ is proper. Fix any $x_0 \in L$. Then $\Gamma(x_0) \subset L$. By the properness of the map I, $\Gamma_{\tilde{S}}\backslash\Gamma(x_0)$ is a finite subset of $\Gamma_{\tilde{S}}\backslash L$. On the other hand, $\Gamma_{x_0} = \{e\}$ since Γ is torsion-free. Therefore $\Gamma_{\tilde{S}}\backslash\Gamma$ is finite, yielding a contradiction. $\square$

6. Orbital counting for discrete hyperbolic groups

6.1. **Hyperboloid model of $\mathbb{H}^n$.** We refer to [34] for basic facts on hyperbolic geometry. For the quadratic form

$$Q_0(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2,$$

the hyperboloid model of $\mathbb{H}^n$ is given by

$$\mathbb{H}^n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : Q_0(x) = -1, x_{n+1} > 0\}$$

where the metric is induced from the Minkowski metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$, and the distance $d(x, y)$ between $x = (x_1, \cdots, x_{n+1})$ and $y = (y_1, \cdots, y_{n+1})$ is given by $\cosh d(x, y) = -(x_1y_1 + \cdots + x_ny_n) + x_{n+1}y_{n+1}$. Set G to be the identity component of $\mathrm{SO}_{Q_0}(\mathbb{R})$, $o := e_{n+1}^t = (0, \cdots, 0, 1)^t$, $\mathrm{SO}(n) = \{g \in \mathrm{SL}_n(\mathbb{R}) : g^t g = I_n\}$ and $K := \{\left[\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}\right] \in \mathrm{SL}_{n+1}(\mathbb{R}) : g \in \mathrm{SO}(n)\}$. Since G acts transitively on $\mathbb{H}^n$ and the stabilizer of o in G is equal to K, we have $\mathbb{H}^n = G.o \cong G/K$. The group G acts on $G/K = \mathbb{H}^n$ as isometries, via the left translations: $L(g_0)(gK) = (g_0g)K$. This induces an action of G on the unit tangent bundle $\mathrm{T}^1(\mathbb{H}^n)$ given by $g(p, v) = (g.p, dL(g)_p(v))$ for $(p, v) \in \mathrm{T}^1(\mathbb{H}^n)$; here $v \in \mathrm{T}_p(G/K)$ is a unit vector based at p and $dL(g)_p : \mathrm{T}_p(\mathbb{H}^n) \to \mathrm{T}_{g(p)}(\mathbb{H}^n)$ is the differential of $L(g)$ at p. For the following discussion, we refer to [3, II.4]. Set $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{k} = \mathrm{Lie}(K)$. Consider the Cartan involution $\theta(g) := (g^t)^{-1}$ of G and the corresponding positive definite symmetric bilinear form on $\mathfrak{g}$: $B_\theta(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(\theta(Y)))$. We have $\mathfrak{k} = \{X \in \mathfrak{g} : \theta(X) = X\}$, and the following orthogonal decomposition of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ where $\mathfrak{p}$ is the -1 eigenspace of θ. The map $\mathfrak{p} \times K \to G$ given by $(X, k) \mapsto (\exp X)k$ is a diffeomorphism. For the canonical projection $\pi : G \to G/K$, the kernel of $d\pi$ is $\mathfrak{k}$ and hence $d\pi : \mathfrak{p} \to \mathrm{T}_o(G/K)$ is a K-equivariant linear isomorphism where the action of K on $\mathfrak{p}$ is via the adjoint representation.

Set

$$a_r = \left[\begin{smallmatrix} \cosh r & 0 & \sinh r \\ 0 & I_{n-1} & 0 \\ \sinh r & 0 & \cosh r \end{smallmatrix}\right], \quad A^+ := \{a_r : r \geq 0\}, \quad A^- := \{a_r : r \leq 0\},$$

$$A := A^+ \cup A^- \text{ and } M = \{\left[\begin{smallmatrix} 1 & & \\ & g & \\ & & 1 \end{smallmatrix}\right] \in K : g \in \mathrm{SO}(n-1)\}.$$

The Lie algebra $\mathfrak{a}$ of A is a maximal abelian subalgebra of $\mathfrak{p}$, M is equal to the centralizer of $\mathfrak{a}$ in K, and we have the Cartan decomposition $G = KA^+K$

in the sense that $k_1ak_2 = k_1'a'k_2' \in KA^+K$ implies that $a = a'$ and $k_1 = k_1'm$ and $k_2 = m^{-1}k_2'$ for some $m \in M$. Let

$$\omega_0 = \left[\begin{smallmatrix} -I_2 & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right] \in K. \tag{6.1}$$

Note that $\omega_0 M \omega_0^{-1} = M$ and $\omega_0 a_r \omega_0^{-1} = a_{-r}$ for all $r \in \mathbb{R}$. Letting $X_0 \in \log(A^+)$ be the element of norm 1 the map $g \mapsto gX_0$ induces the G-equivariant homeomorphism $\mathrm{T}^1(G/K) = G.X_0 = G/M$. Via the identification $\mathrm{T}^1(G/K) = G/M$, the geodesic flow $\{g^r\}$ on $\mathrm{T}^1(G/K)$ correspond to the right translations by a_r: $g^r(gM) = ga_rM$.

Notation 6.1. We denote by $N < G$ the expanding horospherical subgroup with respect to the right a_r-action, that is,

$$N := \{g \in G : a_r g a_r^{-1} \to e \quad \text{as } r \to \infty\}.$$

The N-leaves gNM/M correspond to expanding horospheres in $\mathrm{T}^1(G/K) = G/M$. In particular, $N.X_0$ is the expanding horosphere passing through o based at the point $X_0^- = \omega_0[M]$ in the boundary $K/M = \partial(\mathbb{H}^n)$.

6.2. **Some computation with m^{BR}.** Let Γ be a torsion-free non-elementary discrete subgroup of G. We use the same notation as in the section 3.

Via the visual map $\mathrm{T}^1(\mathbb{H}^n) \to \partial(\mathbb{H}^n)$ by $u \to u^+$, K/M is homeomorphic to the boundary $\partial(\mathbb{H}^n)$. Hence the K-invariant measure m_o and the Patterson-Sullivan measure ν_o on $\partial(\mathbb{H}^n)$ can be considered as measures on K/M: $d\nu_o(k) = d\nu_o(kX_0^+)$ and $dm_o(k) = dm_o(kX_0^+)$. In the Iwasawa coordinates $G = KAN$, the map $\mathrm{T}^1(\mathbb{H}^n) \to \partial(\mathbb{H}^n)$, $u \mapsto u^-$ is given by

$$j : G/M = \mathrm{T}^1(\mathbb{H}^n) \to K/M$$

with $j[ka_rn] = kX_0^- = [k\omega_0]$. Moreover for $u = [ka_rn] \in \mathrm{T}^1(\mathbb{H}^n)$,

$$\begin{aligned} \beta_{u^-}(o, \pi(u)) &= \beta_{kX_0^-}(o, ka_rno) = \beta_{X_0^-}(o, a_rno) \\ &= \lim_{t\to\infty} d(o, a_{-t}o) - d(a_rno, a_{-t}o) \\ &= \lim_{t\to\infty} t - d(a_{t+r}na_{-t-r}(a_{t+r}o), o) \\ &= \lim_{t\to\infty} t - d(a_{t+r}o, o) = -r, \end{aligned}$$

as $X_0^- = \lim_{t\to\infty} \pi([M]a_{-t}) = \lim_{t\to\infty} a_{-t}(o)$.

Therefore for $\phi \in C_c(G/M)$, we have

$$\tilde{m}^{\mathrm{BR}}(\phi) = \int_{[k]\in K/M} \int_{j^{-1}[k]} \phi(ka_rn)e^{-\delta r} dndrd\nu_o(kX_0^-). \tag{6.2}$$

Definition 6.2. For a function f on K, we define a function $\mathfrak{R}_f$ on G by

$$\mathfrak{R}_f(a_rnk) = e^{-\delta_\Gamma r} f(k) \tag{6.3}$$

for $a_rnk \in ANK = G$. As the product map $A \times N \times K \to G$ is a diffeomorphism, this is well-defined.

In the following proposition, for an ϵ-neighborhood U_ϵ of e in G, let $\psi_\epsilon \in C(G)$ be a non-negative function with support in U_ϵ and have $\int_G \psi_\epsilon(g)dg = 1$.

Proposition 6.3. *Let $f \in C(M\backslash K)$ and $\Omega \subset K$ a Borel subset with $M\Omega = \Omega$ and $\nu_o(\partial(\Omega^{-1})) = 0$. Then*

$$\lim_{\epsilon\to 0} \tilde{m}^{\mathrm{BR}}(f *_\Omega \psi_\epsilon) = \int_{k\in\Omega^{-1}} f(k^{-1})d\nu_o(kX_0^-)$$

where $f *_\Omega \psi_\epsilon(g) := \int_{k\in\Omega} \psi_\epsilon(gk)f(k)dk$.

Proof. We note that $dg = drdndk$, $g = a_r nk$, is a Haar measure on G.

Note that for some uniform constants $\ell_1, \ell_2 > 0$, we have for all $k \in K$ and for all small $\epsilon > 0$ (see Lemma 6.13),

$$kU_\epsilon \subset U_{\ell_1\epsilon}k \subset (A \cap U_{\ell_2\epsilon})(N \cap U_{\ell_2\epsilon})k(K \cap U_{\ell_2\epsilon}).$$

Set $K_\epsilon := (K \cap U_{\ell_2\epsilon})$, $\Omega_{\epsilon+} = \Omega K_\epsilon$ and $\Omega_{\epsilon-} = \cap_{k\in K_\epsilon} \Omega k$. Then there exists a uniform constant $c > 0$ such that for all $k \in K$ and $g \in U_\epsilon$,

$$(1 - c\cdot\epsilon)\, \mathfrak{R}_{f\cdot\chi_{\Omega_{\epsilon-}}}(k^{-1}) \le \mathfrak{R}_{f\cdot\chi_\Omega}(k^{-1}g) \le (1 + c\cdot\epsilon)\, \mathfrak{R}_{f\cdot\chi_{\Omega_{\epsilon+}}}(k^{-1}).$$

On the other hand, by the assumption that $\nu_o(\partial(\Omega^{-1})) = 0$, for any $\eta > 0$, there exists $\epsilon > 0$ such that

$$\nu_o(\Omega_{\epsilon+}^{-1} - \Omega_{\epsilon-}^{-1}) \le \eta.$$

Therefore

$$\begin{aligned}
\tilde{m}^{\mathrm{BR}}(f *_\Omega \psi_\epsilon) &= \int_{g\in G}\int_{k_0\in\Omega} \psi_\epsilon(gk_0)f(k_0)d(k_0)d\tilde{m}^{\mathrm{BR}}(g) \\
&= \int_{KAN}\int_{k_0\in\Omega} \psi_\epsilon(ka_rnk_0)f(k_0)e^{-\delta r}d(k_0)dndrd\nu_o(k\omega_0) \\
&= \int_{k\in K}\int_{ANK} \psi_\epsilon(ka_rnk_0)f(k_0)\chi_\Omega(k_0)e^{-\delta r}drdndkd\nu_o(k\omega_0) \\
&= \int_{k\in K}\int_{g\in G} \psi_\epsilon(kg)\mathfrak{R}_{f\cdot\chi_\Omega}(g)dgd\nu_o(k\omega_0) \\
&= \int_{k\in K}\int_{g\in G} \psi_\epsilon(g)\mathfrak{R}_{f\cdot\chi_\Omega}(k^{-1}g)dgd\nu_o(k\omega_0) \\
&= \int_{k\in K}\int_{g\in G} \psi_\epsilon(g)(1+O(\epsilon))\mathfrak{R}_{f\cdot\chi_{\Omega_{\epsilon\pm}}}(k^{-1})dgd\nu_o(k\omega_0) \\
&= (1+O(\epsilon))\int_{k\in K} \mathfrak{R}_{f\cdot\chi_{\Omega_{\epsilon\pm}}}(k^{-1})d\nu_o(k\omega_0)
\end{aligned}$$

as $\int_G \psi_\epsilon dg = 1$. Since

$$\int_{k\in K} \mathfrak{R}_{f\cdot\chi_{\Omega_{\epsilon\pm}}}(k^{-1})d\nu_o(k\omega_0) = (1+O(\eta))\int_{k\in\Omega^{-1}} f(k^{-1})d\nu_o(k\omega_0),$$

this proves the claim. ∎

6.3. **Counting in sectors.** Let $w_0 \in \mathbb{R}^{n+1}$ be one of

$$e_1 = (1, 0, \cdots, 0),\ e_{n+1} = (0, \cdots, 0, 1),\ e_1 + e_{n+1} = (1, 0, \cdots, 0, 1).$$

Observe that $Q_0(e_1) = 1, \quad Q_0(e_{n+1}) = -1, \quad Q_0(e_1 + e_{n+1}) = 0$, and that

$$G_{e_1} = \left\{ \begin{pmatrix} 1 & \\ & g \end{pmatrix} : g \operatorname{diag}(I_{n-1}, -1)\, g^t = \operatorname{diag}(I_{n-1}, -1) \right\} =: H$$

$$G_{e_{n+1}} = K \quad \text{and} \quad G_{e_1 + e_{n+1}} = NM$$

where G acts on $\mathbb{R}^{n+1}$ by the multiplication from the right.

The generalized Cartan decomposition and the Iwasawa decomposition of G give

$$G = HAK = KA^+K = NMAK \tag{6.4}$$

where the A^+ and the A components of $g \in G$ are uniquely determined and the right K-component is uniquely determined up to the left multiplications by elements of M.

Lemma 6.4. (1) *The orbit $H.o = H/H \cap K$ is a totally geodesic subspace of $\mathbb{H}^n = G/K$, and $H.X_0 = H/M$ describes the set of vectors in $\mathrm{T}^1(\mathbb{H}^n) = G/M$ which are based at $H.o$, orthogonal to $H.o$ and have the same orientation as X_0.*

(2) *The orbit $K.X_0 = K/M$ represents the set $\mathrm{T}^1_o(\mathbb{H}^n)$ of unit tangent vectors at o.*

(3) *The orbit $(NM)X_0 = NM/M$ represents the expanding horosphere $\mathcal{H}^+$ in $\mathrm{T}^1(\mathbb{H}^n)$ passing through o and based at $X_0^- \in K/M = \partial(\mathbb{H}^n)$.*

Proof. As the intersection $H \cap K$ is equal to M which is a maximal compact subgroup of H, the orbit $H.o$ is a totally geodesic subspace of $\mathbb{H}^n = G/K$. The subgroup H is the fixed points of the involution $\sigma : G \to G$ given by

$$\sigma(g) := \begin{pmatrix} -1 & \\ & I_n \end{pmatrix} g \begin{pmatrix} -1 & \\ & I_n \end{pmatrix}.$$

We can check that σ commutes with the Cartan involution θ. With $\mathfrak{h} := \operatorname{Lie}(H)$, we have the following orthogonal decomposition with respect to B_θ: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ into ± 1 eigenspaces of σ. The tangent space $\mathrm{T}_o(H.o)$ is identified with $\mathfrak{h} \cap \mathfrak{p}$ and, since $\mathfrak{a} \subset \mathfrak{q} \cap \mathfrak{p}$ and $\mathfrak{h}$ is orthogonal to $\mathfrak{a}$, the vector $X_0 \in \mathfrak{a}$, as a unit tangent vector at o, is orthogonal to the tangent space $\mathrm{T}_o(H.o)$. Hence the orbit $H.X_0 = H/M$ describes the set of vectors in $\mathrm{T}^1(G/K) = G/M$ which are based at $H.o$ and orthogonal to $H.o$. (2) and (3) have already been observed. □

If α denotes the simple root for the $\operatorname{ad}(\mathfrak{a})$-action on $\mathfrak{g}$, then $\alpha(\log a_t) = t$ and has multiplicity $(n-1)$. The weight space decomposition of $\mathbb{R}^{n+1}$ for the right action of G via the standard representation of G to SL_{n+1} is given by

$$\mathbb{R}^{n+1} = W_\alpha \oplus W_0 \oplus W_{-\alpha}$$

where $W_\alpha = \mathbb{R}(e_1 + e_{n+1})$, $W_0 = \sum_{2\le i\le n} \mathbb{R}e_i$ and $W_{-\alpha} = \mathbb{R}(e_1 - e_{n+1})$. We denote by v^α the projection of v onto W_α. In view of (6.4), we define a sector $B_T(\Omega)$ in w_0G in the following way:

Definition 6.5. For a fixed norm $\|\cdot\|$ on $\mathbb{R}^{n+1}$ and a Borel subset $\Omega \subset K$ with $M\Omega = \Omega$, we set

$$B_T(\Omega) = \begin{cases} \{w \in e_1 A\Omega : \|w\| < T\} \\ \{w \in e_{n+1} A^+\Omega : \|w\| < T\} \\ \{w \in (e_1 + e_{n+1})A\Omega : \|w\| < T\} \end{cases}$$

accordingly.

We define the following subsets of $\mathrm{T}^1(\mathbb{H}^n) = G/M$:

$$\tilde{E} := \begin{cases} H.X_0 \cup H.(-X_0) & \text{for } w_0 = e_1 \\ K.X_0 & \text{for } w_0 = e_{n+1} \\ (NM).X_0 & \text{for } w_0 = e_1 + e_{n+1} \end{cases}$$

and $E = \mathbf{p}(\tilde{E})$. We then have $\mu_{\tilde{E}}^{\mathrm{PS}}$, $\mu_{\tilde{E}}^{\mathrm{Leb}}$, μ_E^{PS} and μ_E^{Leb} accordingly as in Definition 3.5. In the case of $w_0 = e_1$, we further set $\tilde{E}_\pm = H.(\pm X_0)$, $E_\pm = \mathbf{p}(\tilde{E}_\pm)$, and $\mu_{E\pm}^{\mathrm{PS}}$ and $\mu_{E\pm}^{\mathrm{Leb}}$ to be the restrictions of μ_E^{PS} and μ_E^{Leb} to $E_\pm$ respectively. For $w_0 = e_{n+1}$ or $e_1 + e_{n+1}$, we will sometimes denote by $\tilde{E}_+$ and E_+ for $\tilde{E}$ and E respectively, to make notations uniform.

Theorem 6.6. *Suppose that $|m_\Gamma^{\mathrm{BMS}}| < \infty$. Let $w_0 = e_1, e_{n+1}$ or $e_1 + e_{n+1}$. We assume that $w_0\Gamma$ is discrete, and that $|\mu_E^{\mathrm{PS}}| < \infty$. Let $\Omega \subset K$ be a Borel subset such that $M\Omega = \Omega$ and $\nu_o(\partial(\Omega^{-1})) = 0$. Then*

$$\#(w_0\Gamma \cap B_T(\Omega)) \sim c(w_0, \Omega) \cdot T^{\delta_\Gamma},$$

where $0 \le c(w_0, \Omega) < \infty$ is defined as follows: set $\hat{e} := \frac{e_1+e_{n+1}}{2}$ and $\check{e} := \frac{e_1-e_{n+1}}{2}$,

$$c(e_1, \Omega) = \frac{|\mu_{E+}^{\mathrm{PS}}|}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|} \int_{k\in\Omega^{-1}} \|\hat{e}k^{-1}\|^{-\delta} d\nu_o(k(X_0^-)) + \frac{|\mu_{E-}^{\mathrm{PS}}|}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|} \int_{k\in\Omega^{-1}\omega_0} \|\check{e}k^{-1}\|^{-\delta} d\nu_o(k(X_0^-));$$

$$c(e_{n+1}, \Omega) = \frac{|\mu_E^{\mathrm{PS}}|}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|} \int_{k\in\Omega^{-1}} \|\hat{e}k^{-1}\|^{-\delta} d\nu_o(k(X_0^-))$$

and

$$c(e_1 + e_{n+1}, \Omega) = \frac{|\mu_E^{\mathrm{PS}}|}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|} \int_{k\in\Omega^{-1}} \|2\hat{e}k^{-1}\|^{-\delta} d\nu_o(k(X_0^-)).$$

6.4. **Deduction of Theorem 1.4 from Theorem 6.6.** In fact we will prove the following stronger statement:

Theorem 6.7. *Let $0 \neq w_0 \in \mathbb{R}^{n+1}$ be such that $w_0\Gamma$ is discrete. Suppose that $|m_\Gamma^{\mathrm{BMS}}| < \infty$ and $|\mu_E^{\mathrm{PS}}| < \infty$. Then for any norm $\|\cdot\|$ on $\mathbb{R}^{n+1}$,*

$$\#\{v \in w_0\Gamma : \|v\| < T\} \sim \frac{b(w_0)}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot T^{\delta_\Gamma},$$

where $0 \leq b(w_0) = b(w_0, \|\cdot\|) < \infty$ is described in the following paragraph.

6.4.1. *Description of $b(w_0)$.* By the standard facts about the hyperboloid model of $\mathbb{H}^n$ (cf. [11, Sec2]), the E_{w_0} defined in the introduction is equal to E. Given $0 \neq w_0 \in \mathbb{R}^{n-1}$, fix $o \in \pi(\tilde{E}_{w_0})$. Consider any two dimensional linear subspace containing o and w_0. It intersects $\{Q(v) = 0\}$ in two distinct lines, say $[\xi^-]$ and $[\xi^+]$, such that $w_0 \notin [\xi^-]$ (in the case of $Q(w_0) = 0$). Let $\check{w}_0$ and $\hat{w}_0$ denote the orthogonal projection of w_0 on $[\xi^+]$ and on $[\xi^-]$ respectively with respect to the bilinear form Q. Note that if $Q(w_0) > 0$ then the pair $[\xi^\pm]$ is uniquely determined up to a permutation; if $Q(w_0) < 0$ then $o = w_0/\sqrt{Q(w_0)}$ and the pair $[\xi^\pm]$ is not uniquely determined. If $Q(w_0) = 0$ then $\check{w}_0 = w_0$ and the pair is uniquely determined. For $Q(w_0) > 0$ we have $\tilde{E}_{w_0} = \tilde{E}_+ \cup \tilde{E}_-$, where the vectors in $\tilde{E}_\pm$ have the same orientation and their visual image on the boundary is in the same hemisphere containing $[\xi^\pm] \subset \partial\mathbb{H}^n$. We put $\mathrm{sk}_\Gamma(w_0)^\pm = |\mu_{E\pm}^{\mathrm{PS}}|$. We now set $b(w_0)$ as follows: if $Q(w_0) > 0$,

$$b(w_0) = \mathrm{sk}_\Gamma(w_0)^+ \cdot \int_{k\in G_o} \|\hat{w}_0 k\|^{-\delta_\Gamma}\, d\nu_o([\xi^-]k) + \mathrm{sk}_\Gamma(w_0)^- \cdot \int_{k\in G_o} \|\check{w}_0 k\|^{-\delta_\Gamma}\, d\nu_o([\xi^-]k);$$

and if $Q(w_0) \leq 0$,

$$b(w_0) = \mathrm{sk}_\Gamma(w_0) \cdot \int_{k\in G_o} \|\hat{w}_0 k\|^{-\delta_\Gamma}\, d\nu_o([\xi^-]k).$$

Now using the Witt theorem we can deduce Theorem 6.7 from Theorem 6.6 for the case $\Omega = K$. Theorem 1.6 follows from Theorems 4.14, 4.11, and Proposition 5.5.

6.5. **Proof of Theorem 6.6.** This subsection is entirely devoted to the proof of Theorem 6.6. We begin by observing the following:

Lemma 6.8. *If $w_0\Gamma$ is discrete, then the canonical projection map $\Gamma_{w_0}\backslash\tilde{E}_+ \to \mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ is proper and injective with image E_+.*

Proof. Since $w_0\Gamma$ is discrete in $w_0 G = G_{w_0}\backslash G$, $\Gamma\backslash\Gamma G_{w_0}$ is closed in $\Gamma\backslash G$ by [33], and hence the claim on the properness follows from Lemma 6.9 below. Since $G_{w_0} = G_{\tilde{E}_+}$, the injectivity follows from Lemma 2.3. $\square$

The following lemma is a consequence of the Baire category theorem:

Lemma 6.9. *Let $H < G$ be locally compact second countable topological groups and $\Gamma < G$ a closed subgroup. The canonical projection map $H \cap \Gamma \backslash H \to \Gamma \backslash G$ is proper if and only if ΓH is closed in G.*

Note that $\tilde{E} = G_{w_0}/M$ and recall the measure

$$d\mu^{\mathrm{Leb}}_{\tilde{E}}(v) = e^{(n-1)\beta_{v^+}(o,\pi(v))} dm_o(v^+).$$

Without loss of generality, we normalize the conformal density $\{m_x\}$ so that m_o is the probability measure. We denote by dh and dk the Haar measures on H and K given by $d\mu^{\mathrm{Leb}}_{\tilde{E}} dm$ where dm denotes the probability Haar measure on M. On N, the measure $d\mu^{\mathrm{Leb}}_{\tilde{E}}$ for $\tilde{E} = NM/M$ gives the Haar measure dn. For the case $\tilde{E} = K/M$, since $\pi(v) = o$ for all $v \in \tilde{E}$, we have $d\mu^{\mathrm{Leb}}_{\tilde{E}}(v) = dm_o(v)$ and hence dk is the Haar probability measure on K. Via the decompositions $G = HAK, KA^+K, NAK$, we consider the following Haar measure dg on G respectively:

$$2^{n-1} \left(\sinh \alpha(\log a_r) \cdot \cosh \alpha(\log a_r)\right)^{(n-1)/2} dhdrdk \text{ for } g = ha_rk$$
$$2^{n-1} \left(\sinh \alpha(\log a_r) \cdot \cosh \alpha(\log a_r)\right)^{(n-1)/2} dk_1drdk_2 \text{ for } g = k_1a_rk_2$$
$$e^{(n-1)r} dndrdk \text{ for } g = na_rk$$

where dr is the Lebesgue measure on $\mathbb{R}$.

Define the following counting function on $\Gamma \backslash G$:

$$F_{B_T(\Omega)}(g) := \sum_{\gamma \in \Gamma_{w_0} \backslash \Gamma} \chi_{B_T(\Omega)}(w_0 \gamma g).$$

We note $F_{B_T(\Omega)}(e) = \# w_0 \Gamma \cap B_T(\Omega)$. For simplicity, for $w_0 = e_1, e_{n+1}$, we set

$$\Xi(a_r) = 2^{n-1} \left(\sinh(\alpha(\log a_r)) \cdot \cosh(\alpha(\log a_r))\right)^{(n-1)/2},$$

in which case $\Xi(a_r) \sim e^{(n-1)|r|}$ as $r \to \pm\infty$, and for $w_0 = e_1 + e_{n+1}$, we set

$$\Xi(a_r) = e^{(n-1)r}.$$

For $\psi_1, \psi_2 \in C_c(\Gamma \backslash G)$, we set $\langle \psi_1, \psi_2 \rangle := \int_{\Gamma \backslash G} \psi_1(g) \psi_2(g) \; dg$.

Lemma 6.10. *For any $\psi \in C_c(\Gamma \backslash G)$, we have*

$$\langle F_{B_T(\Omega)}, \psi \rangle = \int_{k \in \Omega} \int_{\|w_0 a_r k\| < T} \Xi(a_r) \int_{v \in E_+} \psi_k(v a_r) d\mu^{\mathrm{Leb}}_{\tilde{E}}(v) dr dk$$

where $\psi_k \in C_c(\Gamma \backslash G)^M$ is given by $\psi_k(g) = \int_{m \in M} \psi(gmk)$.

Proof. Writing dx for the Haar measures dh, dk, dn on G_{w_0} accordingly, we deduce

$$\begin{aligned}
&\langle F_{B_T(\Omega)}, \psi\rangle = \int_{\Gamma_{w_0}\backslash G} \chi_{B_T(\Omega)}(w_0 g)\psi(g)dg \\
&= \int_{k\in\Omega}\int_{\|w_0 a_r k\|<T}\int_{\Gamma_{w_0}\backslash G_{w_0}} \psi(x a_r k)\,\Xi(a_r)dxdrdk \\
&= \int_{k\in\Omega}\int_{\|w_0 a_r k\|<T}\int_{v\in\Gamma_{w_0}\backslash\tilde{E}_+}\int_{m\in M} \psi(vma_rk)\,\Xi(a_r)d\mu^{\mathrm{Leb}}_{\tilde{E}}(v)dmdrdk \\
&= \int_{k\in\Omega}\int_{\|w_0 a_r k\|<T}\int_{v\in E_+}\left(\int_{m\in M}\psi(va_rmk)\,dm\right)\Xi(a_r)d\mu^{\mathrm{Leb}}_{\tilde{E}}(v)drdk \\
&= \int_{k\in\Omega}\int_{\|w_0 a_r k\|<T}\Xi(a_r)\int_{E_+}\psi_k(va_r)d\mu^{\mathrm{Leb}}_{\tilde{E}}(v)drdk.
\end{aligned}$$

$\square$

Definition 6.11. Let $f \in C(M\backslash K)$ and $\Psi \in C_c(\Gamma\backslash G)$. For $\Omega \subset K$ with $M\Omega = \Omega$, define the following function in $C_c(\Gamma\backslash G)^M$:

$$f *_\Omega \Psi(g) := \int_{k\in\Omega}\Psi(gk)f(k)dk.$$

In the following proposition, we define, for $v = w_0^{\pm\alpha}$, a function $\xi_v \in C(M\backslash K)$ by

$$\xi_v(k) := \|vk\|^{-\delta_\Gamma}.$$

Proposition 6.12. *Keep the same assumption as in Theorem 6.6. Let $\psi \in C_c(\Gamma\backslash G)$. As $T \to \infty$, we have the following:*

- *For $w_0 = e_1$,*

$$\langle F_{B_T(\Omega)}, \psi\rangle \sim \frac{T^{\delta_\Gamma}}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|}\Big(|\mu^{\mathrm{PS}}_{E+}|\cdot m^{\mathrm{BR}}(\xi_{e_1^\alpha} *_\Omega \psi) + |\mu^{\mathrm{PS}}_{E-}|\cdot m^{\mathrm{BR}}(\xi_{e_1^{-\alpha}} *_{\omega_0\Omega} \psi)\Big);$$

- *For $w_0 = e_{n+1}$ or $e_1 + e_{n+1}$,*

$$\langle F_{B_T(\Omega)}, \psi\rangle \sim \frac{|\mu^{\mathrm{PS}}_E|\cdot T^{\delta_\Gamma}}{\delta_\Gamma\cdot|m^{\mathrm{BMS}}|}m^{\mathrm{BR}}(\xi_{w_0^\alpha} *_\Omega \psi).$$

Proof. We first consider the case when $w_0 = e_1$.

Recalling the definition of ω_0 from (6.1), we set $\psi_+ = \psi$ and $\psi_- = \psi_{\omega_0}$. We claim as $r \to \infty$,

$$e^{(n-1-\delta)r}\int_{v\in E_\pm}\psi(va_r)d\mu^{\mathrm{Leb}}_E(v) \sim \frac{|\mu^{\mathrm{PS}}_{E_\pm}|}{|m^{\mathrm{BMS}}|}m^{\mathrm{BR}}(\psi_\pm). \tag{6.5}$$

For E_+, the claim follows from Theorem 3.10 and Lemma 6.4. Since $\tilde{E}_- = H.(-X_0)$, note that for each $r \in \mathbb{R}$, $\tilde{E}_+ a_{-r} = \tilde{E}_- a_r \omega_0^{-1}$. As ψ is M-invariant

and $\omega_0 M = M\omega_0$, as $r \to \infty$,

$$e^{(n-1-\delta)r}\int_{v\in E_-}\psi(va_{-r})d\mu^{\mathrm{Leb}}_{E_-}(v) = e^{(n-1-\delta)r}\int_{s\in E_-}\psi_{\omega_0^{-1}}(va_r)d\mu^{\mathrm{Leb}}_{E_-}(v)$$
$$\sim \frac{|\mu^{\mathrm{PS}}_{E_-}|}{|m^{\mathrm{BMS}}|}m^{\mathrm{BR}}(\psi_{\omega_0}) \quad \text{proving (6.5) for } E_-.$$

Fixing $\epsilon > 0$ and $k \in K$, by (6.5), there exists $r_0 > 0$ such that for any $r > r_0$,

$$e^{(n-1-\delta)r}\int_{v\in E_\pm}\psi_k(va_r)d\mu^{\mathrm{Leb}}_{E_\pm}(v) = (1+O(\epsilon))\frac{|\mu^{\mathrm{PS}}_{E_\pm}|\cdot m^{\mathrm{BR}}(\psi_{\pm k})}{|m^{\mathrm{BMS}}|}.$$

We may also assume that for $|r| > r_0$.

$$\Xi(a_r) = (1+O(\epsilon))e^{(n-1)|r|}. \tag{6.6}$$

On the other hand, $e_1 = e_1^{\alpha} + e_1^{-\alpha}$ where $e_1^{\alpha} = \frac{e_1+e_{n+1}}{2}$ and $e_1^{-\alpha} = \frac{e_1-e_{n+1}}{2}$. Since $e_1 a_r k = e^r e_1^{\alpha}k + e^{-r}e_1^{-\alpha}k$ and

$$\int_{e^r\|e_1^{\alpha}k\|<T-O(e^{-r_0}),r>r_0} e^{\delta r}dr \le \int_{\|e_1a_rk\|<T,r>r_0} e^{\delta r}dr \le \int_{e^r\|e_1^{\alpha}k\|<T+O(e^{-r_0}),r>r_0} e^{\delta r}dr$$

we have

$$\int_{a_r:\|e_1a_rk\|<T,r>r_0} e^{\delta r}dr = \frac{T^\delta}{\delta\cdot\|e_1^{\alpha}k\|^\delta} + O(e^{r_0}).$$

Similarly, we can deduce

$$\int_{a_r:\|e_1a_rk\|<T,r<-r_0} e^{-\delta r}dr = \frac{T^\delta}{\delta\cdot\|e_1^{-\alpha}k\|^\delta} + O(e^{r_0}).$$

Hence

$$\int_{\|w_0a_rk\|<T,|r|>r_0}\Xi(a_r)\int_{E_+}\psi_k(va_r)d\mu^{\mathrm{Leb}}_{E_+}(v)dr$$
$$= \int_{\|w_0a_rk\|<T,|r|>r_0}\Xi(a_r)e^{(-n+1+\delta)|r|}\left(e^{(n-1-\delta)|r|}\int_{E_+}\psi_k(va_r)d\mu^{\mathrm{Leb}}_{E_+}(v)\right)dr$$
$$= \frac{(1+O(\epsilon))}{|m^{\mathrm{BMS}}|}$$
$$\left(|\mu^{\mathrm{PS}}_{E_+}|\, m^{\mathrm{BR}}(\psi_k)\cdot\int_{\|w_0a_rk\|<T,r>r_0} e^{\delta r}dr + |\mu^{\mathrm{PS}}_{E_-}|\, m^{\mathrm{BR}}(\psi_{\omega_0k})\cdot\int_{\|w_0a_rk\|<T,r<-r_0} e^{-\delta r}dr\right)$$
$$= \frac{(1+O(\epsilon))T^\delta}{\delta|m^{\mathrm{BMS}}|}\left(\frac{|\mu^{\mathrm{PS}}_{E_+}|\, m^{\mathrm{BR}}(\psi_k)}{\|e_1^{\alpha}k\|^\delta} + \frac{|\mu^{\mathrm{PS}}_{E_-}|\, m^{\mathrm{BR}}(\psi_{\omega_0k})}{\|e_1^{-\alpha}k\|^\delta}\right) + O(e^{r_0}).$$

Since $E_+ \subset \mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ is a closed subset, $\psi \in C_c(\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n))$ and K is compact, it follows that for fixed $r_0 > 1$, we have

$$\sup_{k\in K, v\in E_+, |r|<r_0}|\psi_k(va_r)| = O(1).$$

Hence

$$\int_{\|w_0 a_r k\|<T, |r|\le r_0} \Xi(a_r) \int_{E_+} \psi_k(va_r) d\mu_E^{\mathrm{Leb}}(v) dr = O(e^{(n-1)r_0}).$$

Therefore

$$\int_{\|w_0 a_r k\|<T} \Xi(a_r) \int_{E_+} \psi_k(va_r) d\mu_{E_+}^{\mathrm{Leb}}(s) dr$$
$$= O(e^{(n-1)r_0}) + \frac{(1+O(\epsilon))T^\delta}{\delta |m^{\mathrm{BMS}}|} \left(\frac{|\mu_{E_+}^{\mathrm{PS}}| \cdot m^{\mathrm{BR}}(\psi_k)}{\|e_1^\alpha k\|^\delta} + \frac{|\mu_{E_-}^{\mathrm{PS}}| \cdot m^{\mathrm{BR}}(\psi_{\omega_0 k})}{\|e_1^{-\alpha} k\|^\delta} \right).$$

As $T \to \infty$, we have

$$\int_{\|w_0 a_r k\|<T} \Xi(a_r) \int_{E_+} \psi_k(va_r) d\mu_{E_+}^{\mathrm{Leb}}(v) dr$$
$$= \frac{(1+O(\epsilon)) m^{\mathrm{BR}}(\psi_k) T^\delta}{\delta |m^{\mathrm{BMS}}|} \left(\frac{|\mu_{E_+}^{\mathrm{PS}}| \cdot m^{\mathrm{BR}}(\psi_k)}{\|e_1^\alpha k\|^\delta} + \frac{|\mu_{E_-}^{\mathrm{PS}}| \cdot m^{\mathrm{BR}}(\psi_{\omega_0 k})}{\|e_1^{-\alpha} k\|^\delta} \right).$$

As $\epsilon > 0$ is arbitrary, it follows that as $T \to \infty$,

$$\int_{\|w_0 a_r k\|<T} \Xi(a_r) \int_{E_+} \psi_k(va_r) d\mu_{E_+}^{\mathrm{Leb}}(v) dr$$
$$\sim \frac{m^{\mathrm{BR}}(\psi_k) T^\delta}{\delta |m^{\mathrm{BMS}}|} \left(\frac{|\mu_{E_+}^{\mathrm{PS}}| \cdot m^{\mathrm{BR}}(\psi_k)}{\|e_1^\alpha k\|^\delta} + \frac{|\mu_{E_-}^{\mathrm{PS}}| \cdot m^{\mathrm{BR}}(\psi_{\omega_0 k})}{\|e_1^{-\alpha} k\|^\delta} \right).$$

As K is compact, we can integrate this over $k \in \Omega$ and, since $e_1^{-\alpha}\omega_0 = -e_1^{-\alpha}$, we deduce that (6.10) is asymptotic to

$$\frac{T^\delta}{\delta \cdot |m^{\mathrm{BMS}}|} \left(|\mu_{E_+}^{\mathrm{PS}}| m^{\mathrm{BR}}(\xi_{e_1^\alpha} *_\Omega \psi) + |\mu_{E_-}^{\mathrm{PS}}| m^{\mathrm{BR}}(\xi_{e_1^{-\alpha}} *_{\omega_0 \Omega} \psi) \right)$$

as $T \to \infty$. This proves the proposition for $w_0 = e_1$.

For $w_0 = e_{n+1}$, the integral over the set $\{a_r : \|w_0 a_r k\| < T\}$ in (6.10) occurs only for $r \ge 0$ due to the decomposition $G = G_{e_{n+1}} A^+ K$. In view of this, the same argument as above yields the claim in the proposition.

Now consider the case of $w_0 = e_1 + e_{n+1}$. Then $\|w_0 a_r k\| = e^r \|w_0 k\|$. One can show similarly as before that the integral over $\{a_r : \|w_0 a_r k\| < T\}$ in (6.10) is dominated by the integral over $\{a_r : 1 \ll e^r \le \frac{T}{\|w_0 k\|}\}$ in (6.10). The rest of the proof is similar to the case of $w_0 = e_1$, and hence we omit. $\square$

Lemma 6.13 (Strong wavefront lemma)**.** [14, Thm 4.1] *There exists $\ell > 1$ and $\epsilon_0 > 0$ such that for any ϵ-neighborhood U_ϵ of G with $\epsilon < \epsilon_0$ and for any $g = hak \in HAK$ with $\|a\| > 2$,*

$$gU_\epsilon \subset h(H \cap U_{\ell\epsilon}) a(A \cap U_{\ell\epsilon}) k(K \cap U_{\ell\epsilon}).$$

Similar statements hold for KA^+K and NAK-decompositions.

Proof of Theorem 6.6: By the assumption that $\nu_o(\partial(\Omega^{-1}))=0$, for all sufficiently small $\epsilon>0$, there exists an ϵ-neighborhood K_ϵ of e in K such that for $\Omega_{\epsilon+}=\Omega K_\epsilon$ and $\Omega_{\epsilon-}=\cap_{k\in K_\epsilon}\Omega k$,

$$\lim_{\epsilon\to 0}\nu_o(\Omega_{\epsilon+}^{-1}-\Omega_{\epsilon-}^{-1})=0. \tag{6.7}$$

By Lemma 6.13, with $\ell_0:=\ell^{-1}<1$ as therein, we have for all $T\gg 1$,

$$B_T(\Omega)U_{\ell_0\epsilon}\subset B_{(1+\epsilon)T}(\Omega_{\epsilon+})\quad\text{and}\quad B_{(1-\epsilon)T}(\Omega_{\epsilon-})\subset\cap_{u\in U_{\ell_0\epsilon}}B_T(\Omega)u.$$

Clearly, we can assume that $U_\epsilon K$ injects to $\Gamma\backslash G$. Let $\psi_\epsilon\in C_c(G)$ be a non-negative function supported on $U_{\ell_0\epsilon}$ and $\int\psi_\epsilon dg=1$, and let $\Psi_\epsilon\in C_c(\Gamma\backslash G)$ the Γ-average of ψ_ϵ:

$$\Psi_\epsilon(g):=\sum_{\gamma\in\Gamma}\psi_\epsilon(\gamma g).$$

We then have for all $g\in U_{\ell_0\epsilon}$

$$F_{B_{(1-\epsilon)T}(\Omega_{\epsilon-})}(g)\le F_{B_T(\Omega)}(e)\le F_{B_{(1+\epsilon)T}(\Omega_{\epsilon+})}(g).$$

Therefore, by integrating against Ψ_ϵ, we have

$$\langle F_{B_{(1-\epsilon)T}(\Omega_{\epsilon-})},\Psi_\epsilon\rangle\le F_{B_T(\Omega)}(e)\le\langle F_{B_{(1+\epsilon)T}(\Omega_{\epsilon+})},\Psi_\epsilon\rangle.$$

By Proposition 6.3, for any $\eta>0$, there exists $\epsilon>0$ such that

$$m^{\mathrm{BR}}(\xi_{w_0^{\pm\alpha}}*_\Omega\Psi_\epsilon)=\tilde m^{\mathrm{BR}}(\xi_{w_0^{\pm\alpha}}*_\Omega\psi_\epsilon)=(1+O(\eta))\cdot\int_{k\in\Omega^{-1}}\|w_0^{\pm\alpha}k^{-1}\|^{-\delta}d\nu_o(k).$$

Letting $w_0=e_1$, Proposition 6.12 now implies that

$$\limsup_T T^{-\delta_\Gamma}\cdot\langle F_{B_{(1\pm\epsilon)T}(\Omega_{\epsilon\pm})},\Psi_\epsilon\rangle=\frac{(1+O(\eta))}{\delta\cdot|m^{\mathrm{BMS}}|}$$
$$\left(|\mu^{\mathrm{PS}}_{E_+}|\int_{k\in\Omega_{\epsilon\pm}^{-1}}\|e_1^\alpha k^{-1}\|^{-\delta}d\nu_o(k\omega_0)+|\mu^{\mathrm{PS}}_{E_-}|\int_{k\in\omega_0\Omega_{\epsilon\pm}^{-1}}\|e_1^{-\alpha}k^{-1}\|^{-\delta}d\nu_o(k\omega_0)\right)$$
$$=\liminf_T T^{-\delta_\Gamma}\cdot\langle F_{B_{(1\pm\epsilon)T}(\Omega_{\epsilon\pm})},\Psi_\epsilon\rangle.$$

Using (6.7), it is now easy to deduce that $F_{B_T(\Omega)}(e)\sim c_0(e_1,\Omega)\cdot T^\delta$. This finishes the proof for $w_0=e_1$. For the other cases of $w_0=e_{n+1},e_1+e_{n+1}$, the proof is similar.

6.6. **Counting in cones.** Consider the Euclidean norm on $\mathbb{R}^{n+1}$. Let $\tilde\Omega$ be a Borel subset of the unit sphere $S^n=\{x_1^2+\cdots+x_{n+1}^2=1\}$ and $\mathcal{C}=\mathcal{C}(\tilde\Omega)$ be the cone spanned by $\tilde\Omega$, that is, $\mathcal{C}=\mathbb{R}_{\ge 0}\tilde\Omega$. Let $w_0=e_1,e_{n+1}$ or e_1+e_{n+1}. Let

$$v_\infty=\lim_{t\to\infty}\frac{w_0a_t}{\|w_0a_t\|}=(e_1+e_{n+1})/2.$$

Let $\Theta=\{\theta\in\mathrm{SO}(n+1):v_\infty\theta\in\tilde\Omega\}$ and $\Omega=\Theta\cap K$.

Theorem 6.14. *Suppose that* $|m_\Gamma^{\mathrm{BMS}}| < \infty$ *and* $|\mu_E^{\mathrm{PS}}| < \infty$. *Assume* $\nu_o(\partial(\Omega^{-1}) \cup \partial(\Omega^{-1}\omega_0)) = 0$. *Then*

$$\lim_{T\to\infty} \frac{\#\{v \in w_0\Gamma \cap \mathcal{C}(\tilde{\Omega}) : \|v\| < T\}}{T^{\delta_\Gamma}} = c(w_0, \Omega).$$

where $c(w_0, \Omega) \geq 0$ *is defined as follows:*

$$c(e_1, \Omega) = \frac{\sqrt{2}^\delta}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|}(|\mu_{E+}^{\mathrm{PS}}|\nu_o(\Omega^{-1}(X_0^-)) + |\mu_{E-}^{\mathrm{PS}}|\nu_o(\Omega^{-1}(X_0^+)));$$

$$c(e_{n+1}, \Omega) = \frac{\sqrt{2}^\delta |\mu_E^{\mathrm{PS}}|}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|}\nu_o(\Omega^{-1}(X_0^-));$$

$$c(e_1 + e_{n+1}, \Omega) = \frac{|\mu_E^{\mathrm{PS}}|}{\sqrt{2}^\delta \cdot \delta_\Gamma \cdot |m^{\mathrm{BMS}}|}\nu_o(\Omega^{-1}(X_0^-)).$$

Sketch of the proof. The proof of this result is almost same as the proof of Theorem 6.6. The only difference is the following additional argument. Let $v_t = \frac{w_0 a_t}{\|w_0 a_t\|} \in S^{n+1}$. Let $\Omega_t = \{k \in K : v_t k \in \tilde{\Omega}\}$. Let $\theta_t \in \mathrm{SO}(n+1)$ such that $v_t = w_0\theta_t$. Then $\Omega_t = M\theta_t^{-1}\Omega \cap K = \theta_t^{-1}\Theta \cap K$. Since $v_t \to v_\infty$, we choose θ_t so that $\theta_t \to e$ as $t \to \infty$. Let K_ϵ denote an ϵ-neighborhood of e in K. Then for sufficiently small $\epsilon > 0$, for all large $t > 0$, we have $\theta_t^{-1}\Theta K_\epsilon \subset \Theta K_{2\epsilon}$ and $\cap_{k\in K_{2\epsilon}}\Theta k \subset \theta_t^{-1}\Theta K_\epsilon$. Hence $\cap_{k\in K_{2\epsilon}}\Omega k \subset \Omega_t K_\epsilon \subset \Omega K_{2\epsilon}$. Since $\nu_o(\partial\Omega^{-1}) = 0$, we have that $\nu_o((\Omega K_{2\epsilon} - \cap_{k\in K_{2\epsilon}}\Omega k)^{-1}) \to 0$ as $\epsilon \to 0$. This well roundedness property of Ω_t allows one to do the counting in the above case, and the final result correspond to the case of Theorem 6.6 with Ω as above. □

7. Hyperbolic and Spherical Apollonian circle packings

We follow the notations set up in the introduction 1.5. Let $\mathcal{P}$ be an integral Euclidean, hyperbolic or spherical Apollonian circle packing where all the circles C in $\mathcal{P}$ are labeled according to the corresponding curvatures $\mathrm{curv}_\star(C)$ where $\star$ stands for $E(clidean)$, $H(yperbolic)$ and $S(pherical)$. Fix a period $\mathcal{P}_0$ of $\mathcal{P}$ so that

$$N_{\mathcal{P}}^\star(T) := \#\{C \in \mathcal{P}_0 : \mathrm{Curv}_\star(C) < T\} < \infty$$

for any $T > 1$. We will deduce the following from Theorem 1.4:

$$N_{\mathcal{P}}^\star(T) \sim c \cdot T^\alpha \tag{7.1}$$

for some $c = (\mathcal{P}, \star)$, where α is the residual dimension of $\mathcal{P}$.

Consider the Descartes quadratic form:

$$Q(x_1, x_2, x_3, x_4) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2.$$

The quadruple of curvatures $\mathrm{Curv}_\star(C_1)$, $\mathrm{Curv}_\star(C_2)$, $\mathrm{Curv}_\star(C_3)$, $\mathrm{Curv}_\star(C_4)$ of 4 mutually tangent circles in $\mathcal{P}$ satisfies the equation

$$Q(x_1, x_2, x_3, x_4) = q(\star)$$

where $q(E) = 0$, $q(H) = 4$ and $q(S) = -4$. This is the Descartes circle theorem for $\star = E$, the spherical Soddy-Gossett theorem, and the hyperbolic Soddy-Gossett theorem for $\star = S, H$ respectively (see [20]).

The Apollonian group $\mathcal{A} < \mathrm{GL}_4(\mathbb{Z})$ is generated by the following four reflections S_1, S_2, S_3, S_4 respectively:

$$\begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

The Apollonian group $\mathcal{A}$ is a subgroup of $\mathrm{O}_Q(\mathbb{Z})$ of infinite index and its critical exponent $\delta_{\mathcal{A}}$ is equal to α.

The group $\mathcal{A}$ acts transitively on the set $\mathfrak{D} = \mathcal{D}(\star)$ of all Descartes quadruples (a, b, c, d) associated to $\mathcal{P}$, that is, the quadruples of curvatures of all 4 mutually tangent circles of $\mathcal{P}$. Hence $\mathfrak{D} = \xi_0(\star)\mathcal{A}^t$ for $\xi_0(\star) \in \mathcal{D}(\star)$.

We can deduce in the same manner as in [19] using the results in [10] that

$$N_{\mathcal{P}}^{\star}(T) = \#\{v \in \xi_0(\star)\mathcal{A}^t : \|v\|_{\max} < T\},$$

up to a fixed additive constant. Set G to be the identity component of $\mathrm{SO}_Q(\mathbb{R})$. The subgroup $\mathcal{A}^t \cap G$ is of finite index in $\mathcal{A}^t$ and hence is a non-elementary geometrically finite group with $\delta_{\mathcal{A}^t \cap G}$ being precisely equal to α, and the orbit $\xi_0(\star)(\mathcal{A}^t \cap G) \subset \mathbb{Z}^4$ is discrete in $\mathbb{R}^4$. Hence (7.1) follows from Theorem 1.6, using $\delta_{\mathcal{A}^t \cap G} > 1$. It is also possible prove that $\xi_0(\star)$ is not $\mathcal{A}^t \cap G$-external, which will then provide a proof of (7.1) independent of the fact $\delta_{\mathcal{A}^t \cap G} > 1$. In the two figures Fig. 2 and Fig. 3, this claim can be directly seen from the configurations. As the general case requires a quite involved argument using several theorems of [10], we'll omit it.

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